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Wallman covers of compact spaces
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§ 1. Introduction

Throughout this paper all hypothesized topological spaces are assumed to be completely regular and Hausdorff (i.e. Tikhonov). Thus the word "space" will mean "completely regular Hausdorff topological space".

A compactification of a space $X$ is a pair consisting of a compact space $K$ and a dense embedding map $i: X \to K$. Usually we view $X$ as a dense subspace of $K$ and let $i$ be the inclusion map.

A class of compactifications of Tikhonov spaces that has been widely studied is the class of Wallman compactifications. Such a compactification is constructed by adjoining to $X$ the set of all ultrafilters on a certain ring $\mathcal{C}$ of closed subsets of $X$ (called a Wallman base for $X$), and topologizing the resulting set. The fact that the resulting space $\omega(X)$ is Hausdorff and contains $X$ as a dense subspace follows from certain combinatorial and topological assumptions made about the ring $\mathcal{C}$. Wallman compactifications have been widely studied, and most "naturally occurring" compactifications of a space $X$ are, in fact, of the form $\omega(X)$ for a suitably chosen Wallman base $\mathcal{C}$. Indeed, it was not until 1977 that examples of compactifications of spaces $X$ that are not of the form $\omega(X)$ were produced (see [Ba] and [UI]).

Recall that a ring of closed subsets of a space $X$ is a collection of closed sets that contains $\emptyset$ and $X$ is closed under finite unions and intersections. Thus a ring of closed subsets is just a sublattice of the lattice of closed sets (partially ordered by inclusion).

1.1. Definition. Let $\mathcal{S}$ be a collection of closed subsets of a space $X$.

(a) $\mathcal{S}$ is $T_1$ if, whenever $S \in \mathcal{S}$ and $x \in X \setminus S$, there exists $T \in \mathcal{S}$ such that $x \in T$ and $S \cap T = \emptyset$.

(b) If $A, B \in \mathcal{S}$ and $A \cap B = \emptyset$ implies that there exist $C, D \in \mathcal{S}$ with $A \cap C = B \cap D = \emptyset$ and $C \cup D = X$, then $\mathcal{S}$ is normal.

(c) A Wallman base for a space $X$ is a normal $T_1$ ring of closed subsets of a space $X$ that is a base for the closed subsets of $X$.

The research of the third-named author was partially supported by Grant No. A7592 from the Natural Sciences and Engineering Research Council of Canada.
(d) Let $\mathcal{C}$ be a Wallman base for a space $X$ and let $w_\mathcal{C}(X)$ denote the set of ultrafilters (i.e. maximal filters) on the lattice $\mathcal{C}$. If $C \in \mathcal{C}$ let $C^* = \{ x \in w_\mathcal{C}(X): C \in x \}$. One can prove that $\{ C^*: C \in \mathcal{C} \}$ is a closed base for a compact Hausdorff topology on $w_\mathcal{C}(X)$ and that the map $x \to \{ C \in \mathcal{C}: x \in C \}$ embeds $X$ as a dense subspace of $w_\mathcal{C}(X)$ (see 19L, 19M of [Wi] or § 4.4 of [PW] for details and further references). The space $w_\mathcal{C}(X)$ is called the Wallman compactification of $X$ associated with the Wallman base $\mathcal{C}$.

(e) A compactification $\alpha X$ of a space $X$ is called a Wallman compactification of $X$, or is said to be of Wallman type, if there is a Wallman base $\mathcal{C}$ for $X$ such that $\alpha X$ and $w_\mathcal{C}(X)$ are equivalent compactifications of $X$ (Recall $\alpha X$ and $\delta X$ are equivalent compactifications of $X$ if there is a homeomorphism $h: \alpha X \to \delta X$ such that $h(x) = x$ for each $x \in X$.)

(f) A compactification $\alpha X$ of a space $X$ is called regular Wallman if there is a Wallman base $\mathcal{C}$ for $X$ consisting of regular closed subsets of $X$ (i.e. sets that are closures of open sets) such that $\alpha X$ and $w_\mathcal{C}(X)$ are equivalent compactifications of $X$.

Recall (see Chapter 3 of [PW], or [Wo]) that the collection $\mathcal{R}(X)$ of all regular closed subsets of a space $X$, when partially ordered by inclusion, becomes a complete Boolean algebra, in which the sup, inf, and complementation operations are defined as follows: if $A \in \mathcal{R}(X)$ and $\{ A_\alpha: \alpha \in \Lambda \} \subseteq \mathcal{R}(X)$ then

$$\bigvee \{ A_\alpha: \alpha \in \Lambda \} = \text{cl}_X(\bigcup \{ A_\alpha: \alpha \in \Lambda \})$$

$$\bigwedge \{ A_\alpha: \alpha \in \Lambda \} = \text{cl}_X \text{int}_X(\bigcap \{ A_\alpha: \alpha \in \Lambda \})$$

$$A' = \text{cl}_X(X \setminus A)$$

1.2. Definition. (a) A subalgebra of $\mathcal{R}(X)$ is a subset of $\mathcal{R}(X)$ that contains $\emptyset$ and $X$, is closed under the formation of finite suprema and infima, and is closed under complementation. (Thus a “subalgebra” of $\mathcal{R}(X)$ is a (not necessarily complete) Boolean algebra with respect to the same operations as those used on $\mathcal{R}(X)$).

(b) A sublattice of $\mathcal{R}(X)$ is a subset of $\mathcal{R}(X)$ that contains $\emptyset$ and $X$ and is closed under the formation of finite suprema and infima. (Thus a “sublattice” of $\mathcal{R}(X)$ is a (not necessarily complete) lattice with respect to the same lattice operations as those used on $\mathcal{R}(X)$. It need not be closed under complementation).

A notion dual to the concept of a compactification of a space $X$ is the concept of a cover of a space $X$. In what follows, the covering map from a space onto $X$ is analogous to the embedding of $X$ as a dense subset of a compact space.
1.3. Definition. (a) A function \( f \) from a space \( Y \) onto a space \( X \) is called a covering map if it is a perfect irreducible continuous surjection. (Recall that the surjection \( f: Y \to X \) is irreducible if it maps proper closed subsets of \( Y \) onto proper closed subsets of \( X \).)

(b) A cover of a space \( X \) is a pair \( (Y, f) \) where \( Y \) is a space and \( f: Y \to X \) is a covering map.

An extensive discussion of covers and covering maps, and a development of the duality between compactifications (and other extensions) and covers, appears in § 8.4 of [PW]. The best-known example of a cover of a space is its absolute; see [Wo] and Chapter 6 of [PW] for a detailed treatment of this notion.

We will make use of the following result; (a) appears as 2.3 of [Wo] and as 6.5(a) of [PW]; and (b) appears in [H] and in 8.4(i) of [PW].

1.4. Theorem. (a) If \( (Y, f) \) is a cover of \( X \) then \( A \mapsto f[A] \) is a Boolean algebra isomorphism from \( \mathcal{R}(Y) \) onto \( \mathcal{R}(X) \).

(b) If \( h, k, \) and \( m \) are covering maps and \( h \circ k = h \circ m \) then \( k = m \).

1.5. Definition. Let \( (Y, f) \) be a cover of \( X \).

(a) If \( \mathcal{A} \) is a subset of \( \mathcal{R}(Y) \), denote by \( \mathcal{A}(X, f) \) the set \( \{ f[A] : A \in \mathcal{A} \} \).

(b) If \( \mathcal{D} \) is a subset of \( \mathcal{R}(X) \), denote by \( \mathcal{D}^{-}(Y, f) \) the set \( \{ \text{cl}_{Y} f^{-}[\text{int}_{X} D] : D \in \mathcal{D} \} \).

The next result is an immediate consequence of 1.4.

1.6. Theorem. Let \( (Y, f) \) be a cover of \( X \).

(a) If \( \mathcal{A} \) is a subalgebra (resp. sublattice) of \( \mathcal{R}(Y) \) then \( \mathcal{A}(X, f) \) is a subalgebra (resp. sublattice) of \( \mathcal{R}(X) \).

(b) If \( \mathcal{D} \) is a subalgebra (resp. sublattice) of \( \mathcal{R}(X) \) then \( \mathcal{D}^{-}(Y, f) \) is a subalgebra (resp. sublattice) of \( \mathcal{R}(Y) \).

(c) If \( \mathcal{A} \) and \( \mathcal{D} \) are as above then
\[
\mathcal{A}(X, f)^{-}(Y, f) = \mathcal{A} \quad \text{and} \quad (\mathcal{D}^{-}(Y, f))(X, f) = \mathcal{D}.
\]

Now let \( X \) denote a compact space. In § 2 we introduce the concept of a Wallman sublattice of \( \mathcal{R}(X) \); its definition is analogous to the definition of a Wallman base (see 1.1). Corresponding to each Wallman sublattice \( \mathcal{A} \) we construct a "smallest" cover \( (\mathcal{L}(\mathcal{A}, X), \psi_{\mathcal{A}}) \) of \( X \) which has properties dual to those of the Wallman compactification of \( X \) corresponding to a Wallman base. Such a cover will be called a Wallman cover of \( X \). Many of the standard "covers" appearing in the literature are specific instances of this general construction (see § 2 for details). In § 3 we study the Wallman cover of \( X \) corresponding to the Wallman sublattice of \( \mathcal{R}(X) \) consisting of the closures of "complemented cozero-sets" of \( X \). (A cozero-set \( C \) of a space \( X \) is complemented if there exists another cozero-set \( W \) of \( X \) such that \( C \cap W = \emptyset \) and \( C \cup W \) is dense in \( X \).) In § 4 we use known results on Wallman
compactifications to show that there exist spaces not all of whose covers are Wallman covers.

This paper uses the terminology of [GJ] and [PW], to which we refer the reader puzzled by undefined terms.

§ 2. The construction of the Wallman cover

Throughout this section $X$ denotes a fixed compact Hausdorff space. As noted in § 1, $(\mathcal{R}(X), \leq)$ is a complete Boolean algebra and hence, in particular, a complete lattice. The following definitions are analogous to those in 1.1.

2.1. DEFINITION. Let $\mathcal{A}$ be a sublattice of $\mathcal{R}(X)$.

(a) $\mathcal{A}$ is said to be $T_1$ with respect to $X$ if for each $A \in \mathcal{A}$ and $x \in X \setminus A$ there exists $B \in \mathcal{A}$ such that $x \in \text{int}_X B$ and $A \cap B = \emptyset$ (recall $A \cap B = \text{cl}_X \text{int}_X (A \cap B)$; see § 1).

(b) $\mathcal{A}$ is said to be normal if, whenever $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$, there exist $C, D \in \mathcal{A}$ such that $A \cap C = B \cap D = \emptyset$ and $C \cup D = X$. [Note that without loss of generality one can assume that $A \subseteq D$ and $B \subseteq C$.]

(c) If $\mathcal{A}$ satisfies (a) and (b) it is called a Wallman sublattice of $\mathcal{R}(X)$.

Note the similarity between (b) of the above definition and condition 1.1 (b) in the definition of a Wallman base. There is also some resemblance between 2.1 (a) and 1.1 (a). This is our motivation for the choice of terminology in 2.1 (c). Note that the definition of a Wallman sublattice is not purely lattice-theoretic, as it depends on the topology of $X$ as well as the lattice-theoretic properties of $\mathcal{R}(X)$. However, we do have the following:

2.2. PROPOSITION. Let $(Y, f)$ be a cover of $X$ and let $\mathcal{A}$ be a sublattice of $\mathcal{R}(Y)$. If $\mathcal{A}$ is $T_1$ with respect to $Y$ then $\mathcal{A}(X, f)$ is $T_1$ with respect to $X$.

Proof. Let $A \in \mathcal{A}$ and $x \in X \setminus f[A]$. Then $f^{-1}(x) \cap A = \emptyset$, so for each $y \in f^{-1}(x)$ there exists $B(y) \in \mathcal{A}$ such that $y \in \text{int}_Y B(y)$ and $A \cap B(y) = \emptyset$. By the compactness of $f^{-1}(x)$ there exist $y_1, \ldots, y_n \in f^{-1}(x)$ such that $f^{-1}(x) \subseteq \bigcup_{i=1}^n \text{int}_Y B(y_i).$ Then $\bigcup_{i=1}^n B(y_i) = B \in \mathcal{A}$. Thus $f^{-1}(x) \cap B' = \emptyset$ and so $f[A] \cap B' = (f[B])'$ (see 1.4). Thus $x \in X \setminus (f[B])' = \text{int}_X f[B]$. As $A \cap B(y_i) = \emptyset$ for each $i$ it follows that $A \cap B = \emptyset$, so by 1.4 $f[A] \cap f[B] = \emptyset$. As $f[B] \in \mathcal{A}(X, f)$, our result follows. $
$ Note that the definition of a normal sublattice of $\mathcal{R}(X)$ is purely lattice-theoretic. Precisely:

2.3 PROPOSITION. Let $(Y, f)$ be a cover of $X$ and $\mathcal{A}$ a sublattice of $\mathcal{R}(Y)$. Then $\mathcal{A}$ is a normal sublattice of $\mathcal{R}(Y)$ iff $\mathcal{A}(X, f)$ is a normal sublattice of $\mathcal{R}(X)$. 

2.4. DEFINITION. Let \( X \) be a space and \( \mathcal{A} \) a sublattice of \( \mathcal{H}(X) \). We denote the set of ultrafilters on \( \mathcal{A} \) by \( \mathcal{L}(\mathcal{A}) \). If \( A \in \mathcal{A} \), define \( A^* \) to be \( \{ x \in \mathcal{L}(\mathcal{A}) : A \in x \} \).

We now topologize \( \mathcal{L}(\mathcal{A}) \). The following result is part of 2.1 of [HGW], to which we refer the reader for a proof.

2.5. THEOREM. Let \( \mathcal{A} \) be a sublattice of \( \mathcal{H}(X) \). Then:
(a) \( A \) is a filter on \( \mathcal{A} \) is an ultrafilter on \( \mathcal{A} \) iff, for each \( A \in \mathcal{A} \), there exists \( B \in \mathcal{A} \) such that \( A \cup B = \emptyset \).
(b) \( \{ A^* : A \in \mathcal{A} \} \) is a closed base for a compact topology \( \tau \) on \( \mathcal{L}(A) \).
(c) If \( \mathcal{A} \) is a normal sublattice of \( \mathcal{H}(X) \), then \( \tau \) is a Hausdorff topology on \( \mathcal{L}(\mathcal{A}) \).

Note that our definition of the topology \( \tau \) on \( \mathcal{L}(\mathcal{A}) \) resembles closely the topology defined on the Wallman compactification \( w_{\mathcal{A}}(X) \) of the space \( X \) associated with the Wallman base \( G \). The proof that \( w_{\mathcal{A}}(X) \) is Hausdorff uses property 1.1(b) of \( G \), and the fact that \( \mathcal{L}(\mathcal{A}) \) is Hausdorff (for \( \mathcal{A} \) normal) uses 2.1(b).

2.6. DEFINITION. (a) Let \( X \) be a compact Hausdorff space and let \( \mathcal{A} \) be a Wallman sublattice of \( \mathcal{H}(X) \). Then \( \mathcal{L}(\mathcal{A}, X) \) will denote the subspace \( \{ (x, y) \in \mathcal{L}(\mathcal{A}) \times X : x \in \bigcap \mathcal{A} \} \) of the product space \( \mathcal{L}(\mathcal{A}) \times X \).
(b) If \( X \) and \( \mathcal{A} \) are as above, denote by \( \psi_{\mathcal{A}} \) the restriction to \( \mathcal{L}(\mathcal{A}, X) \) of the projection map from \( \mathcal{L}(\mathcal{A}) \times X \) onto \( X \).

Let \( X \) and \( \mathcal{A} \) be as in 2.6. Then:

2.7. PROPOSITION. (a) \( \mathcal{L}(\mathcal{A}, X) \) is compact
(b) \( \psi_{\mathcal{A}} : \mathcal{L}(\mathcal{A}, X) \rightarrow X \) is a covering map, and so \( (\mathcal{L}(\mathcal{A}, X), \psi_{\mathcal{A}}) \) is a cover of \( X \).
(c) If \( A \in \mathcal{A} \) then \( \overline{\psi_{\mathcal{A}}[\text{int}_{\mathcal{A}} A]} = (A^* \times X) \cap \mathcal{L}(\mathcal{A}, X) \).

Proof. (a) We show that \( \mathcal{L}(\mathcal{A}, X) \) is a closed subset of \( \mathcal{L}(\mathcal{A}) \times X \) and hence compact (as \( \mathcal{L}(\mathcal{A}) \) is compact by 2.5). Let \( (x, y) \in (\mathcal{L}(\mathcal{A}) \times X) \setminus \mathcal{L}(\mathcal{A}, X) \). Thus \( x \notin \bigcap \mathcal{A} \) so there exists \( F \in \mathcal{A} \) such that \( x \notin F \). Using 2.1(a) choose \( A \in \mathcal{A} \) such that \( x \in \text{int}_{\mathcal{A}} A \) and \( A \cap F = \emptyset \). Then \( x \in \mathcal{L}(\mathcal{A}) \setminus A^* \), and the set \( \mathcal{L}(\mathcal{A}) \setminus A^* \times \text{int}_{\mathcal{A}} A \) is open in \( \mathcal{L}(\mathcal{A}) \times X \), contains \( (x, y) \) and is easily shown to be disjoint from \( \mathcal{L}(\mathcal{A}, X) \). Our result follows.

(b) Since \( \mathcal{L}(\mathcal{A}, X) \) is compact, in order to show that \( \psi_{\mathcal{A}} \) is a covering map it suffices to show that \( \psi_{\mathcal{A}} \) is an irreducible map onto \( X \).

Let \( x \in X \) and define \( \mathcal{F}(x) \) to be \( \{ A \in \mathcal{A} : x \in \text{int}_{\mathcal{A}} A \} \). Then \( \mathcal{F}(x) \) is easily verified to be a filter on \( \mathcal{A} \) and hence is contained in some \( x \in \mathcal{L}(\mathcal{A}) \). We claim that \( x \in \bigcap \mathcal{A} \); for if not, then find \( F \in \mathcal{A} \) for which \( x \notin F \). By 2.1(a) there exists \( A \in \mathcal{A} \) such that \( x \in \text{int}_{\mathcal{A}} A \) and \( A \cap F = \emptyset \). Clearly \( A \in \mathcal{F}(x) \subseteq \bigcap \mathcal{A} \) yet \( A \cap F = \emptyset \), contradicting the fact that \( \mathcal{A} \) is a filter. Thus \( (x, y) \in \mathcal{L}(\mathcal{A}, X) \) and \( \psi_{\mathcal{A}}(x, y) = (A^* \times X) \cap \mathcal{L}(\mathcal{A}, X) \). Hence \( \psi_{\mathcal{A}} \) is surjective.
To show that $\psi_{\mathcal{A}}$ is irreducible, it suffices to show that if $\mathcal{B}$ is a base for the open sets of $\mathcal{L}(\mathcal{A}, X)$ and $B \in \mathcal{B}$, then $\psi_{\mathcal{A}}[\mathcal{L}(\mathcal{A}, X) \setminus B] \neq X$. The sets 

$\{\mathcal{L}(\mathcal{A}, X) \setminus [(\mathcal{L}(\mathcal{A}) \setminus A^*) \times V] : A \in \mathcal{A}, V \text{ open in } X\}$

form an open base for $\mathcal{L}(\mathcal{A}, X)$. Suppose $A \in \mathcal{A}$, $V$ is open in $X$, and $(a, x_0) \in \mathcal{L}(\mathcal{A}, X) \setminus [(\mathcal{L}(\mathcal{A}) \setminus A^*) \times V]$. Then $a \notin A^*$ so $A \notin a$, and by 2.5(a) there exists $F \in a$ such that $A \setminus F = \emptyset$, i.e., $\text{int}_X A \cap \text{int}_X F = \emptyset$. As $(a, x_0) \in \mathcal{L}(\mathcal{A}, X)$ and $F \in a$ it follows that $x_0 \notin F$; since $x_0 \in V$ it follows that $V \cap \text{int}_X F \neq \emptyset$. We claim that $V \cap \text{int}_X F \subseteq X \setminus \psi_{\mathcal{A}}[\mathcal{L}(\mathcal{A}, X) \setminus [(\mathcal{L}(\mathcal{A}) \setminus A^*) \times V]]$; this will show that $\psi_{\mathcal{A}}$ is irreducible. If $t \in V \cap \text{int}_X F$ and $(t, t) \in \mathcal{L}(\mathcal{A}, X)$, then $t \in a$ and so $t \notin a$ as $A \setminus F = \emptyset$. Thus $(t, t) \in (\mathcal{L}(\mathcal{A}) \setminus A^*) \times V$ and so $\psi_{\mathcal{A}}(t) \subseteq (\mathcal{L}(\mathcal{A}) \setminus A^*) \times V$ and our claim follows. Thus $(\mathcal{L}(\mathcal{A}, X), \psi_{\mathcal{A}})$ is a cover of $X$.

(c) First note that if $(a, x) \in \psi_{\mathcal{A}}[\text{int}_X A]$, then $x \in \text{int}_X A \cap (\bigcap a)$, so $(\text{int}_X A) \cap F \neq \emptyset$ for each $F \in a$. Hence $F \cap A \neq \emptyset$ for each $F \in a$, so by 2.5(a) $A \in a$. Thus $a \in A^*$ and $(a, x) \in (A^* \times X) \cap (\mathcal{L}(\mathcal{A}, X))$. Thus cl$_{(\mathcal{L}(\mathcal{A}, X), \psi_{\mathcal{A}})}[\text{int}_X A] \subseteq (A^* \times X) \cap (\mathcal{L}(\mathcal{A}, X))$.

Conversely, if $(a, x) \in \mathcal{L}(\mathcal{A}, X) \setminus \psi_{\mathcal{A}}[\text{int}_X A]$ there exists $B \in \mathcal{A}$ and an open subset $V$ of $X$ such that $(a, x) \in \left[ (\mathcal{L}(\mathcal{A}) \setminus B^*) \times V \right] \cap \mathcal{L}(\mathcal{A}, X)$ and $[(\mathcal{L}(\mathcal{A}) \setminus B^*) \times V] \cap \psi_{\mathcal{A}}[\text{int}_X A] = \emptyset$. As $a \notin B^*$ there exists $G \in a$ with $G \cap B = \emptyset$. Let us assume that $(a, x) \in A^* \times X$; then $A \in a$ so $A \cap G \neq \emptyset$. As $x \in (\bigcap a) \cap V$, it follows that $(\text{int}_X A \cap \text{int}_X G) \cap V \neq \emptyset$. Let $y \in (\text{int}_X A \cap \text{int}_X G) \cap V$, and choose $a \in (\mathcal{L}(\mathcal{A}) \setminus B^*)$ such that $y \in a \cap V$. Thus $(\text{int}_X G) \cap (\bigcap a) \neq \emptyset$, so arguing as above we see that $G \cap a \neq \emptyset$. As $G \cap B = \emptyset$, $B \notin a$ so $(a, y) \in (\mathcal{L}(\mathcal{A}) \setminus B^*) \times V$. However, $(a, y) \notin \psi_{\mathcal{A}}[\text{int}_X A]$ which gives a contradiction. Thus $(a, x) \notin A^* \times X$ and so $(A^* \times X) \cap (\mathcal{L}(\mathcal{A}, X)) \subseteq \text{cl}_{(\mathcal{L}(\mathcal{A}, X), \psi_{\mathcal{A}})}[\text{int}_X A]$. Our result follows.

We can now state and prove the fundamental theorem of this section.

2.8. Theorem. Let $X$ be a compact Hausdorff space and let $\mathcal{A}$ be a Wallman sublattice of $\mathcal{B}(X)$. Then the cover $(\mathcal{L}(\mathcal{A}, X), \psi_{\mathcal{A}})$ has the following properties:

(a) If $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$ then

$$\text{cl}_{(\mathcal{L}(\mathcal{A}, X), \psi_{\mathcal{A}})}[\text{int}_X A] \cap \text{cl}_{(\mathcal{L}(\mathcal{A}, X), \psi_{\mathcal{A}})}[\text{int}_X B] = \emptyset$$

(b) If $F, G \in \mathcal{A}^{-}(\mathcal{L}(\mathcal{A}, X), \psi_{\mathcal{A}})$ then $F \cap G = F \cap G$ (where infima are taken in $\mathcal{B}(\mathcal{L}(\mathcal{A}, X))$).

(c) $\mathcal{A}^{-}(\mathcal{L}(\mathcal{A}, X), \psi_{\mathcal{A}})$ is $T_1$ with respect to $\mathcal{L}(\mathcal{A}, X)$.

(d) The cover $(\mathcal{L}(\mathcal{A}, X), \psi_{\mathcal{A}})$ is the smallest cover of $X$ with respect to property (a) in this sense: if $(Y, f)$ is a cover of $X$ and if $\text{cl}_Y f^{-1}[\text{int}_X A] \cap \text{cl}_Y f^{-1}[\text{int}_X B] = \emptyset$ whenever $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$, then there exists a covering map $g : Y \to \mathcal{L}(\mathcal{A}, X)$ such that $f = \psi_{\mathcal{A}} \circ g$.

Proof. To prove (b), suppose $F, G \in \mathcal{A}^{-}(\mathcal{L}(\mathcal{A}, X), \psi_{\mathcal{A}})$. Then $F = \text{cl}_{(\mathcal{L}(\mathcal{A}, X), \psi_{\mathcal{A}})}[\text{int}_X A]$ and $G = \text{cl}_{(\mathcal{L}(\mathcal{A}, X), \psi_{\mathcal{A}})}[\text{int}_X B]$ for some elements
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\[ A \text{ and } B \text{ of } \mathcal{A}. \text{ Thus} \]
\[ F \cap G = \text{cl}_{\mathcal{L}(\mathcal{A}, X)}[\text{int}_X A] \cap \text{cl}_{\mathcal{L}(\mathcal{A}, X)}[\text{int}_X B] \]
\[ = (A^* \times X) \cap (B^* \times X) \cap \mathcal{L}(\mathcal{A}, X) \]
\[ = [(A^* \cap B^*) \times X] \cap \mathcal{L}(\mathcal{A}, X). \]

But
\[ A^* \cap B^* = \{ x \in \mathcal{L}(\mathcal{A}) : A \in x \text{ and } B \in x \} \]
\[ = \{ x \in \mathcal{L}(\mathcal{A}) : A \land B \in x \} \]
\[ = (A \land B)^*. \]

Thus
\[ F \cap G = [(A \land B)^* \times X] \cap \mathcal{L}(\mathcal{A}, X) \]
\[ = \text{cl}_{\mathcal{L}(\mathcal{A}, X)}[\text{int}_X (A \land B)] \]
(by 2.7(c)).

Hence \( F \cap G \) is the closure of an open set of \( \mathcal{L}(\mathcal{A}, X) \) and hence belongs to \( \mathcal{B}(\mathcal{L}(\mathcal{A}, X)) \). Obviously this means that \( F \cap G \) is the largest member of \( \mathcal{B}(\mathcal{L}(\mathcal{A}, X)) \) that is smaller than both \( F \) and \( G \), and hence \( F \cap G = F \land G \).

Part (a) is obviously a special case of (b).

To prove (c), choose an element \( G \) of \( \mathcal{L}(\mathcal{A}, X, \psi_\mathcal{A}) \); it will be of the form \( \text{cl}_{\mathcal{L}(\mathcal{A}, X)}[\text{int}_X A] \) for some \( A \in \mathcal{A} \). Let
\[ (x, x) \in \mathcal{L}(\mathcal{A}, X) \setminus G = \mathcal{L}(\mathcal{A}, X) \setminus (A^* \times X) \]
(by 2.7(c)).

Thus \( A \notin x \) so there exists \( F \in x \) such that \( A \land F = \emptyset \). Thus by (b) \( A \land F = \emptyset \).

As \( \mathcal{A} \) is a normal sublattice of \( \mathcal{B}(X) \), there exist \( C, D \in \mathcal{A} \) such that \( A \subseteq C \), \( F \subseteq D \), \( C \cup D = X \), and \( A \land D = F \land C = \emptyset \). As \( F \in x \) and \( F \land C = \emptyset \), it follows that \( C \notin x \) and so \( (x, x) \in \mathcal{L}(\mathcal{A}, X) \setminus (C^* \times X) \). Since \( C \cup D = X \) it follows quickly that \( C^* \cup D^* = \mathcal{L}(\mathcal{A}, X) \) and so \( (x, x) \in \mathcal{L}(\mathcal{A}, X) \setminus (C^* \times X) \subset \mathcal{L}(\mathcal{A}, X) \setminus \mathcal{L}(\mathcal{A}, X, \psi_\mathcal{A}) \) and hence \( G \subset \mathcal{L}(\mathcal{A}, X) \setminus \mathcal{L}(\mathcal{A}, X, \psi_\mathcal{A}) \). Thus \( \mathcal{L}(\mathcal{A}, X, \psi_\mathcal{A}) \) is \( T_1 \) with respect to \( \mathcal{L}(\mathcal{A}, X) \). Note that \( D \in x \) as \( F \subseteq D \) and \( F \in x \).

To prove (d), let \((Y, f)\) be as hypothesized, and let \( y \in Y \). Define \( \alpha(y) \) as follows:
\[ \alpha(y) = \{ A \in \mathcal{A} : y \in \text{cl}_{\mathcal{L}(\mathcal{A}, X)}[\text{int}_X A] \}. \]

We will shown that if \( \mathcal{F} \) is a finite subcollection of \( \alpha(y) \), then \( \bigwedge \{ F : F \in \mathcal{F} \} \neq \emptyset \). If this were not the case, there would exist a natural number \( n_0 \), and a collection of \( n_0 + 1 \) members of \( \alpha(y) \) — say \( \{ A_i : i = 1 \text{ to } n_0 + 1 \} \) — such that \( \bigwedge \{ A_i : i = 1 \text{ to } n_0 + 1 \} = \emptyset \) but any collection of \( n_0 \) members of \( \alpha(y) \) has non-empty infimum. Let \( \bigwedge \{ A_i : i = 1 \text{ to } n_0 \} = A \) and \( A_{n_0 + 1} = B \). Then
$A, B \in \mathcal{A}$ and $A \cap B = \emptyset$. As $\mathcal{A}$ is normal there exist $C, D \in \mathcal{A}$ such that $A \subseteq C$, $B \subseteq D$, $C \cup D = X$, and $A \cap D = B \cap C = \emptyset$. By our hypothesis on $(Y, f)$ and the fact that $y \in \text{cl}_y \beta \left( \text{int}_X B \right)$, we see that $y \notin \text{cl}_y \beta \left( \text{int}_X C \right)$. Since $B \subseteq D$ it follows that $D \in \mathcal{A}(y)$. Since $C \cup D = X$, it follows that $\text{int}_X C \cup \text{int}_X D$ is dense in $X$, and the irreducibility of $f$ then implies that $\beta \left( \text{int}_X C \right) \cup \beta \left( \text{int}_X D \right) = U$ is dense in $Y$, as is easily checked. Now $y \in \text{cl}_y \beta \left( \text{int}_X A_i \right)$ for $i = 1$ to $n_0$, so $y \in \text{cl}_y \beta \left( \text{int}_X A_i \right) \cap U$ for $i = 1$ to $n_0$. But

$$\text{cl}_y \beta \left( \text{int}_X A_i \right) \cap U$$

$$= \text{cl}_y \beta \left( \text{int}_X A_i \right) \cap \beta \left( \text{int}_X D \right) \cup \text{cl}_y \beta \left( \text{int}_X A_i \right) \cap \beta \left( \text{int}_X C \right),$$

and as $y \notin \text{cl}_y \beta \left( \text{int}_X C \right)$ it follows that

$$y \in \text{cl}_y \beta \left( \text{int}_X A_i \right) \cap \beta \left( \text{int}_X D \right)$$

$$= \text{cl}_y \beta \left( \text{int}_X (A_i \cap D) \right)$$

for $i = 1$ to $n_0$.

Thus $A_i \cap D \in \mathcal{A}(y)$ by the definition of $\mathcal{A}(y)$ for $i = 1$ to $n_0$; however, $\bigwedge \{ A_i \cap D : i = 1 \text{ to } n_0 \} = A \cap D = \emptyset$ by choice of $D$, which contradicts our definition of $n_0$. Hence any finite subcollection of $\mathcal{A}(y)$ has non-empty infimum as claimed.

An application of Zorn’s lemma quickly shows that $\mathcal{A}(y)$ must be contained in some ultrafilter on $\mathcal{A}$. We will show that $\mathcal{A}(y)$ is contained in a unique ultrafilter. For if $\delta_1$ and $\delta_2$ were distinct ultrafilters containing $\mathcal{A}(y)$ as a subset, there would exist $F_1 \in \delta_1$, $F_2 \in \delta_2$ for which $F_1 \cap F_2 = \emptyset$ (see 2.5(a)). As $\mathcal{A}$ is normal, there exist $C, D \in \mathcal{A}$ such that $F_1 \cap C = F_2 \cap D = \emptyset$, $F_1 \subseteq C$, and $C \cup D = X$. As above, $\beta \left( \text{int}_X C \right) \cup \beta \left( \text{int}_X D \right)$ is dense in $Y$, so $y \in \text{cl}_y \beta \left( \text{int}_X C \right) \cup \text{cl}_y \beta \left( \text{int}_X D \right)$. However, $y \notin \text{cl}_y \beta \left( \text{int}_X C \right)$, for otherwise $C \in \mathcal{A}(y) \subseteq \delta_1$, $F_1 \in \delta_1$, and $F_1 \cap C = \emptyset$, which is a contradiction. Similarly $y \notin \text{cl}_y \beta \left( \text{int}_X D \right)$, and we have a contradiction.

Let $\delta(y)$ be the unique ultrafilter on $\mathcal{A}$ that contains $\mathcal{A}(y)$. We claim that $f(y) \in \bigcap \delta(y)$; for if not, there exists $F \in \delta(y)$ such that $f(y) \notin F$. As $\mathcal{A}$ is $T_1$ with respect to $X$ there exists $A \in \mathcal{A}$ such that $f(y) \in \text{int}_X A$ and $A \cap F = \emptyset$. Thus $y \in \text{cl}_y \beta \left( \text{int}_X A \right)$ and so $A \in \mathcal{A}(y) \subseteq \delta(y)$, contradicting $A \cap F = \emptyset$. Hence $f(y) \in \bigcap \delta(y)$ as claimed.

It follows that $(\delta(y), f(y)) \in \mathcal{L}(\mathcal{A}, X)$ and hence we can define a function $g : Y \to \mathcal{L}(\mathcal{A}, X)$ as follows:

$$g(y) = \left( \delta(y), f(y) \right).$$

It is clear that $\psi_{\mathcal{A}} \circ g(y) = f(y)$ and so $\psi_{\mathcal{A}} \circ g = f$. To show that $g$ is continuous, let $\mathcal{L}(\mathcal{A}, X) \cap \left( \mathcal{L}(\mathcal{A}) \setminus \mathcal{A} \right) \times V$ be a basic open neighborhood of $(\delta(y), f(y))$ in $\mathcal{L}(\mathcal{A}, X)$. Thus $A \notin \delta(y)$ and $f(y) \in V$. Arguing exactly as we did in the proof of (c), we can find $D \in \delta(y)$ such that $(\delta(y), f(y)) \in \text{cl}_{\mathcal{L}(\mathcal{A}, X)} \mathcal{L}(\mathcal{A}, X) \cap \text{cl}_{\mathcal{L}(\mathcal{A}, X)} \mathcal{L}(\mathcal{A}), \mathcal{L}(\mathcal{A}) \setminus \mathcal{A} \times V \cap \mathcal{L}(\mathcal{A}, X) \cap \mathcal{L}(\mathcal{A}) \setminus \mathcal{A} \times V$. By 1.4 it follows that $A \cap D = \emptyset$. As $\mathcal{A}$ is normal there exist $C, D \in \mathcal{A}$ such
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that \( A \subseteq C, D \subseteq E, C \cup E = X, \) and \( E \land A = D \land C = \emptyset. \) Arguing as before, we see that \( f^{-1} \left[ \text{int}_x C \right] \cup f^{-1} \left[ \text{int}_x E \right] \) is dense in \( Y. \) Now \( y \notin \text{cl}_Y \left[ f^{-1} \left[ \text{int}_x C \right] \right], \) for otherwise \( C \in \mathcal{A}(y) \subseteq \delta(y), D \in \delta(y), \) yet \( C \land D = \emptyset. \) Thus \( y \in Y \setminus \text{cl}_Y \left[ f^{-1} \left[ \text{int}_x C \right] \right] \subseteq f^{-1} \left[ \text{int}_x E \right], \) so \( (Y \setminus \text{cl}_Y \left[ f^{-1} \left[ \text{int}_x C \right] \right]) \cap f^{-1} \left[ V \right] \) is a neighborhood \( U \) of \( y \) in \( Y. \) We will show that \( g[U] \subseteq \mathcal{L}(\mathcal{A}, X) \cap \left( \left( [\mathcal{L}(\mathcal{A}) \setminus A^*] \times V \right) \right), \) thereby verifying the continuity of \( g \) at the arbitrarily chosen point \( y. \) If \( t \in U \) then \( E \in \mathcal{A}(t). \) But \( E \land A = \emptyset \) so \( A \notin \delta(t), \) i.e. \( \delta(t) \in \mathcal{L}(\mathcal{A}) \setminus A^*. \) As \( i \in f^{-1} \left[ V \right], \) \( f(t) \in V \) so \( g(t) \in \mathcal{L}(\mathcal{A}, X) \cap \left( [\mathcal{L}(\mathcal{A}) \setminus A^*] \times V \right) \) as claimed. Thus \( g \) is continuous. It then follows from 8.4(d) of [PW] that \( g \) is a covering map. The proof of the theorem is complete. \( \blacksquare \)

In § 8.4 of [PW] two covers (see 1.3) of a space \( X \) — say \((Y, f)\) and \((Z, g)\) — are said to be equivalent if there exists a homeomorphism \( h: Y \to Z \) such that \( f = g \circ h. \) If equivalent covers are identified, then the set \( \mathcal{C}(X) \) of all covers of a space \( X \) is shown to be a complete lattice with respect to the following order: \((Y, f) \leq (Z, g)\) is defined to mean that there is a covering map \( k: Z \to Y \) such that \( g = f \circ k. \) The absolute \((EX, k_X)\) is the largest member of this lattice, and \((X, 1_X)\) is the smallest. Thus the word "smallest", as used in 2.8(d), refers to this partial ordering of covers of \( X. \)

We now consider several special cases of 2.8.

2.9. Theorem. Let \( \mathcal{A} \) be a compact space and let \( \mathcal{A} \) be a Wallman sublattice of \( \mathcal{E}(X). \) The following are equivalent:

(a) \( \psi_{\mathcal{A}} \) is one-to-one and hence a homeomorphism.
(b) \( A \land B = A \land B \) whenever \( A, B \in \mathcal{A}. \)
(c) If \( A, B \in \mathcal{A} \) and \( A \land B = \emptyset \) then \( A \land B = \emptyset. \)

Proof. (a)⇒(b). Let \( A, B \in \mathcal{A}. \) Then

\[
\text{cl}_{\mathcal{E}(\mathcal{A}, X)} \psi_{\mathcal{A}}[\text{int}_X A] \cap \text{cl}_{\mathcal{E}(\mathcal{A}, X)} \psi_{\mathcal{A}}[\text{int}_X B] = \text{cl}_{\mathcal{E}(\mathcal{A}, X)} \psi_{\mathcal{A}}[\text{int}_X A] \land \text{cl}_{\mathcal{E}(\mathcal{A}, X)} \psi_{\mathcal{A}}[\text{int}_X B]
\]

by 2.8(b). By 1.4, when the map \( \psi_{\mathcal{A}} \) is applied to the right-hand side of the above we obtain \( A \land B; \) since \( \psi_{\mathcal{A}} \) is one-to-one, when we apply the map \( \psi_{\mathcal{A}} \) to the left-hand side of the above we obtain \( A \land B. \) Thus (b) follows

(b)⇒(c). This is obvious.

(c)⇒(a). If \( \psi_{\mathcal{A}} \) is not one-to-one there exist \( \delta_1, \delta_2 \in \mathcal{L}(\mathcal{A}) \) and \( x \in X \) such that \( \delta_i, x) \in \mathcal{L}(\mathcal{A}, X) \) (\( i = 1, 2 \)) and \( \delta_1 \neq \delta_2. \) By 2.5(a) there exist \( A \in \delta_1 \) and \( B \in \delta_2 \) such that \( A \land B = \emptyset. \) But \( x \in (\bigcap \delta_1) \cap (\bigcap \delta_2) \subseteq A \land B, \) so \( A \land B \neq \emptyset. \) \( \blacksquare \)

Let \( k \) denote the restriction to \( \mathcal{L}(\mathcal{A}, X) \) of the projection map from \( \mathcal{L}(\mathcal{A}) \times X \) onto \( \mathcal{L}(\mathcal{A}). \) Note that \( k \) is surjective, for if \( \alpha \in \mathcal{L}(\mathcal{A}) \) then \( \alpha \) is a collection of closed subsets of \( X \) with the finite intersection property and hence \( \bigcap \alpha \neq \emptyset. \) If \( x \in \bigcap \alpha \) then \( (\alpha, x) \in \mathcal{L}(\mathcal{A}, X) \) and \( k(\alpha, x) = \alpha. \)
2.10. Theorem. The following are equivalent for a Wallman sublattice $\mathcal{A}$ of $\mathcal{A}(X)$:
(a) $k$ is one-to-one and hence a homeomorphism.
(b) $\mathcal{A}$ is a base for the closed sets of $X$.

Proof. First note that $k$ is one-to-one iff $\bigcap \alpha$ contains precisely one point for each $\alpha \in \mathcal{L}(\mathcal{A})$.

(a) $\Rightarrow$ (b). If $k$ is one-to-one and hence a homeomorphism, then $\psi_\mathcal{A} \circ k^{-1}$ is a continuous surjection (which we will denote by $g$) from $\mathcal{L}(\mathcal{A})$ onto $X$, and if $\alpha \in \mathcal{L}(\mathcal{A})$ then $g(\alpha)$ is the unique point $x_\alpha$ in $\bigcap \alpha$. If $A \in \mathcal{A}$ then $g(A^*) = \{g(\alpha); A \in \alpha\} = x_\alpha \in A$, so $g[A^*] \subseteq A$. Conversely, if $a \in A$ then by Zorn's lemma there is an ultrafilter $\alpha \in \mathcal{L}(\mathcal{A})$ such that $\{B \in \mathcal{A}; x \in \text{int}_x B\} \subseteq x$. If $\bigcap \alpha \neq \{x\}$ then there exists $D \in \alpha$ such that $x \notin D$. As $\mathcal{A}$ is $T_1$ with respect to $X$ there exists $C \in \mathcal{A}$ such that $x \in \text{int}_x C$ and $C \cup D = \emptyset$. Since $C \in \alpha$, this gives a contradiction. We conclude that $\bigcap \alpha = \{x\}$ and so $g(\alpha) = x$. As $\alpha \in A^*$, it follows that $g(A^*) = A$.

If $F$ is closed in $X$ and $p \in X \setminus F$, then $g^{-1}(p)$ and $g^{-1}[F]$ are disjoint compact subsets of $\mathcal{L}(\mathcal{A})$. As $\{A^*; A \in \mathcal{A}\}$ is a closed base for $\mathcal{L}(\mathcal{A})$, a standard compactness argument shows that there exist $A_1, \ldots, A_n \in \mathcal{A}$ such that $g^{-1}[F] \subseteq \bigcap_{i=1}^n A_i^*$ and $\bigcap_{i=1}^n A_i^* \cap g^{-1}(p) = \emptyset$. It is easy to show that $\bigcap_{i=1}^n A_i^* = \left(\bigwedge_{i=1}^n A_i\right)^*$. Thus $F \subseteq \bigwedge_{i=1}^n A_i \in \mathcal{A}$, and $p \notin \bigwedge_{i=1}^n A_i$. It follows that $\mathcal{A}$ is a base for the closed sets of $X$.

(b) $\Rightarrow$ (a). Let $\mathcal{A}$ be a base for the closed subsets of $X$, let $\alpha \in \mathcal{L}(\mathcal{A})$, and suppose $x$ and $y$ were distinct points of $\bigcap \alpha$. There exist disjoint open sets $U$ and $V$ such that $x \in U$ and $y \in V$. A routine compactness argument shows that there exist $A_1, \ldots, A_n \in \mathcal{A}$ such that $X \setminus U \subseteq \bigcap_{i=1}^n A_i$ and $x \notin \bigcap_{i=1}^n A_i$. Evidently, $y \in \text{int}_x \bigwedge_{i=1}^n A_i$, and so $\bigwedge_{i=1}^n A_i \in \alpha$ (for if $B \in \alpha$ then $B \cup \bigwedge_{i=1}^n A_i \neq \emptyset$ since $y \in B$). Thus $x \in \bigwedge_{i=1}^n A_i$, as $x \in \bigcap \alpha$, which is a contradiction. $\blacksquare$

An important special case occurs when $\mathcal{A}$ is a subalgebra of $\mathcal{A}(X)$ (i.e. when $\mathcal{A}$ is closed under complementation). Note that in this case $\mathcal{L}(\mathcal{A})$ is the Stone space of the Boolean algebra $\mathcal{A}$ and the family of clopen subsets of $\mathcal{L}(\mathcal{A})$ is $\{A^*; A \in \mathcal{A}\}$ (since $A^* = \mathcal{L}(\mathcal{A}) \setminus (A^*)^*$).

2.11. Theorem. Let $\mathcal{A}$ be a subalgebra of $\mathcal{A}(X)$. Then:
(a) $\mathcal{A}$ is a Wallman sublattice of $\mathcal{A}(X)$.
(b) If $A \in \mathcal{A}$ then $\text{cl}_{\mathcal{L}(\mathcal{A}, X)}[\text{int}_X A]$ is a clopen subset of $\mathcal{L}(\mathcal{A}, X)$.
(c) If $(Y, f)$ is a cover of $X$ and if $f \circ \text{int}_X [A]$ is clopen in $Y$ whenever $A \in \mathcal{A}$, then there exists a covering map $g: Y \to \mathcal{L}(\mathcal{A}, X)$ such that $f = \psi_\mathcal{A} \circ g$.

Proof. (a) If $A \in \mathcal{A}$ and $x \in X \setminus A$ then $x \in \text{int}_X A'$ and $A' \cup A = \emptyset$. Thus $\mathcal{A}$ is $T_1$ with respect to $X$. If $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$, then $A \cap A' = B \cap B' = \emptyset$ and $A' \cup B' = A' \cup B = (A \cup B)' = X$. Thus $\mathcal{A}$ is normal.

(b) By 2.8(a), $\text{cl}_{\mathcal{L}(\mathcal{A}, X)}[\text{int}_X A] \cup \text{cl}_{\mathcal{L}(\mathcal{A}, X)}[\text{int}_X A] = \emptyset$, while $\mathcal{L}(\mathcal{A}, X)$ $\mathcal{L}(\mathcal{A}, X) = \psi_\mathcal{A}[A] \cup \psi_\mathcal{A}[A'] = \mathcal{L}(\mathcal{A}, X)$.
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(c) Let \((Y, f)\) be as described, let \(A, B \in \mathcal{A}\), and suppose \(A \cap B = \emptyset\). Then \(B \subseteq A'\) and so

\[
\text{cl}_f f^{-1}[\text{int}_X A] \cap \text{cl}_f f^{-1}[\text{int}_X B] \subseteq \text{cl}_f f^{-1}[\text{int}_X A] \cap \text{cl}_f f^{-1}[\text{int}_X A'].
\]

But by hypothesis \(\text{cl}_f f^{-1}[\text{int}_X A]\) is clopen, so if the preceding intersection were non-empty it would follow that \(f^{-1}[\text{int}_X A \cap \text{int}_X A'] \neq \emptyset\), which is a contradiction. Now apply 2.8(d).  ■

2.12. Definition. A cover \((Y, f)\) of a compact space \(X\) is called a Wallman cover of \(X\) if it is equivalent (in the sense defined preceding the proof of 2.6) to a cover of the form \((\mathcal{L}(\mathcal{A}, X), \psi_{\mathcal{A}})\), where \(\mathcal{A}\) is a Wallman sublattice of \(\mathcal{A}\).

Our definition is motivated by the obvious analogy to Wallman compactifications.

In § 4 we will show that there exist compact spaces with covers that are not Wallman covers. However, most of the “well-known” covers that have appeared in the literature are Wallman covers. We turn to a discussion of these.

2.13. Theorem. Let \((Y, f)\) be a cover of the compact space \(X\) and suppose that \(Y\) is zero-dimensional. Then \((Y, f)\) is a Wallman cover of \(X\).

Sketch of proof. Let \(\mathcal{B}(Y)\) denote the collection of clopen subsets of \(Y\); it is obviously a subalgebra of \(\mathcal{R}(Y)\). Hence by 1.6(a) \(\mathcal{B}(Y)(X, f)\) is a subalgebra of \(\mathcal{R}(X)\) that is isomorphic to \(\mathcal{B}(Y)\); we denote it by \(\mathcal{A}\). By 2.11 \(\mathcal{A}\) is a Wallman sublattice of \(\mathcal{R}(X)\).

If \(y \in Y\) it is easy to verify that \(\{ f[B] : y \in B \in \mathcal{B}(Y) \}\) is an ultrafilter \(\alpha(y)\) on \(\mathcal{A}\). Define \(F: Y \to \mathcal{L}(\mathcal{A}, X)\) as follows: \(F(y) = (\alpha(y), f^*(y))\). A straightforward calculation shows that \(F\) is a well-defined homeomorphism for which \(\psi_{\mathcal{A}} \circ F = f\), and so the Wallman cover \((\mathcal{L}(\mathcal{A}, X), \psi_{\mathcal{A}})\) is equivalent to the cover \((Y, f)\). ■

As noted above, many of the standard “covers” of compact spaces previously discussed in the literature are special cases of the construction described in 2.6. The absolute \((EX, k_X)\) of the compact space \(X\) is constructed by letting \(\mathcal{A} = \mathcal{B}(X)\) in 2.8 (see [Wo] or Chapter 6 of [PW]). We consider two other examples in more detail.

2.14. Example. A quasi-F-space is defined to be a Tikhonov space in which dense cozero-sets are \(C^*\)-embedded. These spaces were introduced in [DHH], where it was shown that each compact space \(X\) has a minimal quasi-F cover \((QF(X), \varphi_X)\). ("Minimal" means that if \(K\) is a compact quasi-F-space and \(g: K \to X\) is a covering map, then there is a covering map \(h: K \to QF(X)\) such that \(g = \varphi_X \circ h\).)

Denote the set of zero-sets of \(X\) by \(\mathcal{Z}(X)\), and the family \(\{\text{cl}_x(\text{int}_x Z) : Z \in \mathcal{Z}(X)\}\) by \(\mathcal{Z}(X)^*\). In [HVW] \((QF(X), \varphi_X)\) is constructed for the compact space \(X\) by showing that \(\mathcal{Z}(X)^*\) is a Wallman sublattice of \(\mathcal{R}(X)\) and setting \(\mathcal{A}\) to be \(\mathcal{Z}(X)^*\) in 2.8 above.
A covering map \( f: K \rightarrow L \) is said to be \( \mathcal{T}^\ast \)-irreducible if \( A \rightarrow f[A] \) is a lattice isomorphism from \( \mathcal{T}(K)^\ast \) onto \( \mathcal{T}(L)^\ast \). In [HVW] it is shown that if \( X \) is compact then \( \varphi_X \) is \( \mathcal{T}^\ast \)-irreducible and that \( (QF(X), \varphi_X) \) is the projective cover of \( X \) in the category of compact spaces and \( \mathcal{T}^\ast \)-irreducible mappings.

2.15. Example. A Tikhonov space is basically disconnected if its cozero-sets have open closures. It was shown independently in [V] and in [ZK] that each compact space has a minimal basically disconnected cover \((BD(X), b_x)\) constructed by setting \( \mathcal{A} \) to be the \( \sigma \)-completion of the subalgebra of \( \mathcal{R}(X) \) generated by the family \( \{cl_x C: C \text{ is a cozero-set of } X\} \) in 2.8 above. As basically disconnected spaces are zero-dimensional, it follows from 2.13 that this is indeed a Wallman cover of \( X \).

It is worth noting that it is shown in 2.6 of [HVW] that the sublattice \( \{cl_x C: C \text{ is a cozero-set of } X\} \) of \( \mathcal{R}(X) \) (where \( X \) is a compact space) is not, in general, a normal sublattice and hence is not Wallman.

In each of the examples discussed above, \( \mathcal{L}(\mathcal{A}, X) \) was isomorphic to \( \mathcal{L}(\mathcal{A}) \) (in the sense of 2.10). In § 3 a different sort of example appears.

§ 3. The minimal clopen cozero-complemented cover of a compact space

In this section we introduce a new Wallman cover \( \mathcal{L}(\mathcal{A}, X) \) of a compact space \( X \) in which \( \mathcal{A} \) is not, in general, a base for the closed subsets of \( X \) and hence \( \mathcal{L}(\mathcal{A}, X) \) is not homeomorphic to \( \mathcal{L}(\mathcal{A}) \).

3.1. Definition. Suppose \( X \) is a Tikhonov space.

(a) A cozero-set \( C \) of \( X \) is called a complemented cozero-set if there is a cozero-set \( D \) of \( X \) such that \( C \cap D = \emptyset \) and \( C \cup D \) is dense in \( X \). We call \( D \) a cozero-complement of \( C \), and \( \{C, D\} \) is called a complementary pair of cozero-sets of \( X \).

(b) Let \( \text{cc}(X) = \{C \in \text{coz}(X): C \text{ is a complemented cozero-set of } X\} \).

(c) Let \( \mathcal{F}(X) = \{cl_x C: C \in \text{cc}(X)\} \).

(d) Let \( \mathcal{B}(X) = \{B: B \text{ is a clopen subset of } X\} \).

Note that \( \mathcal{F}(X) \) is a subalgebra of \( \mathcal{R}(X) \).

3.2. Definition. A space \( X \) is said to be clopen cozero-complemented if \( cl_x C \) is open for each \( C \in \text{cc}(X) \), i.e. if \( \mathcal{F}(X) \) is the Boolean algebra \( \mathcal{B}(X) \) of clopen subsets of \( X \). We shall abbreviate "clopen cozero-complemented space" as "cloz-space".

We now give some characterizations of cloz-spaces; these are followed by examples of cloz-spaces. First we need a definition.
3.3 Definition. A subspace $Y$ of Tikhonov space $X$ is said to be 2-embedded in $X$ if each continuous map of $Y$ into a two-element discrete space $\{a, b\}$ has a continuous extension $F: X \to \{a, b\}$.

3.4. Theorem. If $X$ is Tikhonov space, then the following are equivalent:

(a) $X$ is a cloz-space.

(b) Each dense cozero-set of $X$ is 2-embedded in $X$.

(c) If $f \in C(X)$ and $f$ is not a divisor of zero in $C(X)$, then $f = k|f|$ for some $k \in C(X)$.

(d) $\mathcal{B}(X) = \mathcal{B}(X)$.

Proof. Obviously, (a) and (d) are equivalent. We will show that (a) implies (b), (b) implies (c), and (c) implies (a). First note that $f \in C(X)$ fails to be a divisor of zero iff $\coz f$ is dense in $X$.

Suppose (a) holds, $C$ is a dense cozero-set of $X$, and $B$ is a clopen subset of $C$. By 1.1 of [BH], $B$ and $C \setminus B$ are cozero-sets of $X$, and clearly form a complementary pair. By (a), $\text{cl}_X B \subseteq \mathcal{B}(X)$ and $\text{cl}_X (C \setminus B) = X \setminus \text{cl}_X B$. Thus the characteristic function $\chi_B$ on $C$ extends to the (continuous) characteristic function $\chi_{X \setminus \text{cl}_X B}$ on $X$, so (b) holds.

Suppose (b) holds, $f \in C(X)$, and $\coz f$ is dense in $X$. Then $\text{pos} f = \{x \in X: f(x) > 0\}$ and $\text{neg} f = \{x \in X: f(x) < 0\}$ are complementary clopen subsets of $\coz f$. Hence by assumption the continuous function $g: \coz f \to \{-1, 1\}$ for which $g(x) = 1$ if $x \in \text{pos} f$ and $g(x) = -1$ if $x \in \text{neg} f$ has an extension $k \in C(X)$. Clearly, $f = k|f|$, so (c) holds.

Suppose (c) holds and $C$ and $D$ are a complementary pair of cozero-sets of $X$. Then $C \cup D$ is a dense cozero-set of $X$. There exists $g \in C(X)$ such that $0 \leq g \leq 1$ and $C \cup D = \coz g$. Let $f: X \to \mathbb{R}$ be defined as follows. $f(x) = g(x)$ if $x \in C$, $f(x) = -g(x)$ if $x \in D$, and $f(x) = 0$ otherwise. It is routine to verify that $f \in C(X)$ and $\coz f = C \cup D$. Hence $f$ is not a divisor of zero, so by (c) there exists $k \in C(X)$ for which $f = k|f|$. Evidently, $k(x) = 1$ if $x \in C$ and $k(x) = -1$ if $x \in D$. Thus $\text{cl}_X C \subseteq \text{cl}^* \{-1\}$ and $\text{cl}_X D \subseteq \text{cl}^* \{1\}$, so $\text{cl}_X C \cap \text{cl}_X D = \emptyset$. As $\text{cl}_X C \cup \text{cl}_X D = X$, it follows that $\text{cl}_X C$ is open, and so (a) follows. 

Note that (c) above characterizes cloz-spaces purely in terms of the algebraic and order-theoretic structure of $C(X)$.

Theorem 3.4 provides us with a source of examples of cloz-spaces. They are closely related to quasi-$F$-spaces (see 2.14) as follows.

3.5. Corollary. (a) Every quasi-$F$-space is a cloz-space.

(b) Every strongly zero-dimensional cloz-space is a quasi-$F$-space.

Proof. (a) Since $C^*$-embedding is easily seen to imply 2-embedding, (a) follows from the equivalence of 3.4(a) and (b).

(b) It is shown in 5.4 of [DH] that if $X$ is strongly zero-dimensional (i.e. if $\beta X$ is zero-dimensional) and $C(X)$ satisfies 3.4(c) then $X$ is a quasi-$F$-space.
In 5.6 of [DHH] an example is given of a space satisfying 3.4(c) that is not a quasi-$F$-space. Hence there are cloz-spaces that are not quasi-$F$-spaces. As will be seen below, these two classes of spaces have similarities as well as differences. Recall that a subspace $S$ of $X$ is said to be $z$-embedded in $X$ if for each $Z \in \mathcal{Z}(S)$, there exists $Z' \in \mathcal{Z}(X)$ for which $Z = Z' \cap S$. It is shown in 5.1 of [DHH] that $X$ is a quasi-$F$-space iff each of its dense subspaces is $z$-embedded. For cloz-spaces we have:

3.6. Proposition. A Tikhonov space is a cloz-space iff each of its dense $z$-embedded spaces is 2-embedded.

Proof. Since each cozero-set of $X$ is $z$-embedded in $X$ (see 10.7 of [We]), the sufficiency is immediate from 3.4. Conversely, suppose $S$ is a dense $z$-embedded subspace of a cloz-space $X$ and let $f : S \to \{0, 1\}$ be continuous. By the Blair-Hager approximation theorem (see 2.2 of [BH]), there is a cozero-set $T$ of $X$ that contains $S$, and a $g \in C(T)$, such that $|g(x) - f(x)| < \frac{1}{2}$ if $x \in S$. The density of $S$ and the continuity of $g$ on $T$ imply that $g$ cannot attain the value $\frac{1}{2}$ on $T$, so $\{t \in T : g(t) < \frac{1}{2}\} = V_0$ and $\{t \in T : g(t) > \frac{1}{2}\} = V_1$ yield a partition of $T$ into clopen sets. Define $h : T \to \{0, 1\}$ by $h(x) = 0$ if $x \in V_0$ and $h(x) = 1$ if $x \in V_1$. Then $h \in C(T)$ and $h|S = f$. As $X$ is a cloz-space, by 3.4 $h$ has a continuous two-valued extension over $X$. Hence $S$ is 2-embedded in $X$. $\blacksquare$

We can use 3.5 to obtain another property of cloz-spaces. Recall from Chapter 8 of [GJ] that $\nu X$ denotes the Hewitt realcompactification of $X$.

3.7. Corollary. Let $X$ be a Tikhonov space. The following are equivalent:

(a) $X$ is a cloz-space.

(b) $\nu X$ is a cloz-space.

(c) $\beta X$ is a cloz-space.

Proof. It is well known that $C(X)$ and $C(\nu X)$ are isomorphic as lattice-ordered rings, as are $C^*(X)$ and $C^*(\beta X)$ (see Chapters 8 and 6 respectively of [GJ]). Since the cozero-set of $f$ is dense in $X$ iff $f$ is not a divisor of zero in $C(X)$, the equivalence of (a) and (b) follows from the equivalence of 3.4(a) and 3.4(b). A similar argument yields the equivalence of (a) and (c). $\blacksquare$

The main theorem of this section is the following.

3.8. Theorem. Every compact Hausdorff space $X$ has a unique minimal cloz-cover $(E_{ce}(X), z_X)$ with these properties:

(a) $E_{ce}(X)$ is a compact cloz-space.

(b) If $K$ is a compact cloz-space and if there is a covering map $\varphi : K \to X$, then there is a covering map $g : K \to E_{ce}(X)$ such that $\varphi = z_X \circ g$ (see figure 1).

(c) $(E_{ce}(X), z_X)$ is unique in the sense that if $(E'_{ce}(X), z'_X)$ were another pair satisfying (a) and (b), then there is a homeomorphism $h : E'_{ce}(X) \to E_{ce}(X)$ such that $z_X \circ h = z'_X$.

(d) The map $z$ is $\mathcal{Z}^*-\text{irreducible}$. 


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(e) \( A \to z_x^*[A] \) is a lattice (and thus Boolean algebra) isomorphism from \( \mathfrak{Z}(E_{cc}(X)) \) onto \( \mathfrak{Z}(X) \).

Fig. 1

Proof. Since \( \mathfrak{Z}(X) \) is a subalgebra of \( \mathfrak{R}(X) \), it is a Wallman sublattice of \( \mathfrak{R}(X) \) by 2.11(a). Let \( (E_{cc}(X), z_X) \) be the cover \( \{ \mathfrak{Z}(\mathfrak{Z}(X), X), \psi_{\mathfrak{Z}(X)} \} \) supplied by 2.7(b).

We begin by showing that if \( A, B \in \mathfrak{Z}(X) \) and \( A \land B = \emptyset \), then

\[
\text{cl}_{\mathfrak{Z}(X)} \phi_X^{-1}[\text{int}_X A] \cap \text{cl}_{\mathfrak{Z}(X)} \phi_X^{-1}[\text{int}_X B] = \emptyset.
\]

(Here \( \mathcal{Q}(X), \phi_X \) is the minimal quasi-\( F \)-cover described in 2.14 and discussed in detail in [HVW]). There exists \( M \in \text{cc}(X) \) such that \( A = \text{cl}_X M \). Let \( L \) be a complemented cozero-set of \( M \). Since \( \phi_X \) is irreducible and \( M \) is a dense subset of \( \text{int}_X A \), it follows from 1.4 that \( \text{cl}_{\mathfrak{Z}(X)} \phi_X^{-1}[\text{int}_X A] = \text{cl}_{\mathfrak{Z}(X)} \phi_X^{-1}[M] \). It also follows from 1.4 that \( \phi_X^{-1}[M] \) and \( \phi_X^{-1}[L] \) are complementary cozero-sets of \( \mathcal{Q}(X) \). Thus \( \phi_X^{-1}[M] \cup \phi_X^{-1}[L] \) is a dense cozero-set of \( \mathcal{Q}(X) \) and hence (as \( \mathcal{Q}(X) \) is a quasi-\( F \)-space) is \( C^* \)-embedded in \( \mathcal{Q}(X) \). It immediately follows that

\[
\text{cl}_{\mathfrak{Z}(X)} \phi_X^{-1}[M] \cap \text{cl}_{\mathfrak{Z}(X)} \phi_X^{-1}[L] \neq \emptyset,
\]

i.e.

\[
\text{cl}_{\mathfrak{Z}(X)} \phi_X^{-1}[\text{int}_X A] \cap \text{cl}_{\mathfrak{Z}(X)} \phi_X^{-1}[\text{int}_X B] = \emptyset.
\]

Since \( \text{int}_X A \cap \text{int}_X B = \emptyset \) and \( L \cup M \) is dense in \( X \) while \( M \subseteq \text{int}_X A \), it quickly follows that \( \text{int}_X B \cap L \) is dense in \( \text{int}_X B \). From 1.4 it now follows that

\[
\text{cl}_{\mathfrak{Z}(X)} \phi_X^{-1}[\text{int}_X B] \subseteq \text{cl}_{\mathfrak{Z}(X)} \phi_X^{-1}[L].
\]

Hence \( \text{cl}_{\mathfrak{Z}(X)} \phi_X^{-1}[\text{int}_X A] \cap \text{cl}_{\mathfrak{Z}(X)} \phi_X^{-1}[\text{int}_X B] = \emptyset \) as claimed.

It now follows from 2.8(d) that there exists a covering map \( f : \mathcal{Q}(X) \to E_{cc}(X) \) such that \( z_X \circ f = \phi_X \).

Now we can prove (d). As noted just after 2.8 of [HVW], \( \{ \lambda[A] : A \in \mathcal{P}(S)^* \} \supseteq \mathcal{P}(T)^* \) whenever \( (S, \lambda) \) is a cover of the compact space \( T \). (This also can be deduced quickly from 1.4.) Thus \( \{ z_X[A] : A \in \mathcal{P}(E_{cc}(X))^* \} \supseteq \mathcal{P}(X)^* \). Conversely, suppose that \( A \in \mathcal{P}(E_{cc}(X))^* \). From the above there exists \( M \in \mathcal{P}(\mathcal{Q}(X))^* \) such that \( f[M] = A \). As \( \phi_X \) is \( \mathcal{P}^* \)-irreducible (see 2.14), \( \phi_X[M] \in \mathcal{P}(X)^* \). Thus, as observed above, there exists \( D \in \mathcal{P}(E_{cc}(X))^* \) such that \( z_X[D] = \phi_X[M] = z_X[f[M]] \). By 1.4 \( D = f[M](A) \), so \( z_X[A] = \phi_X[M] \in \mathcal{P}(X)^* \). Thus \( \{ z_X[A] : A \in \mathcal{P}(E_{cc}(X))^* \} \subseteq \mathcal{P}(X)^* \) and \( z_X \) is \( \mathcal{P}^* \)-irreducible. Hence (d) is proved.
Next we prove (a). As $X$ is compact, so is $E_{ec}(X)$. Suppose $C \in \text{cc}(E_{ec}(X))$. Let $D$ be a cozero-complement of $C$. One easily verifies that $\text{cl}_{E_{ec}(X)} \text{int}_{E_{ec}(X)} (E_{ec}(X) \setminus D) = \text{cl}_{E_{ec}(X)} C,$ so $\text{cl}_{E_{ec}(X)} C \in \mathcal{Z}(E_{ec}(X))^*$. Similarly $\text{cl}_{E_{ec}(X)} D \in \mathcal{Z}(E_{ec}(X))^*$. By (d) and 1.4 it follows that $z_X [\text{cl}_{E_{ec}(X)} C]$ and $z_X [\text{cl}_{E_{ec}(X)} D]$ are complementary (in $\mathcal{B}(X)$) members of $\mathcal{Z}(X)^*$ and hence are members of $\mathcal{B}(X)$. It now follows from 2.8(a) and 1.4 that $\text{cl}_{E_{ec}(X)} C \cap \text{cl}_{E_{ec}(X)} D = \emptyset$, and hence that $\text{cl}_{E_{ec}(X)} C$ is open. Hence $E_{ec}(X)$ is a cloz-space and (a) is verified.

If $(S, k)$ is a cover of the space $T$ and if $\{C, D\}$ is a complementary pair of cozero-sets of $T$, then $\{k^-[C], k^-[D]\}$ is a complementary pair of cozero-sets of $S$. If $S$ is a cloz-space, then $\text{cl}_S k^-[C]$ is open so $\text{cl}_S k^-[C] \cap \text{cl}_S k^-[D] = \emptyset$. Part (b) now follows from 2.8(d).

If $(E_{ec}(X), z_X)$ were as hypothesized in (c), there would be covering maps $k^*: E_{ec}(X) \to E_{ec}(X)$ and $k^*: E_{ec}(X) \to E_{ec}(X)$ such that $z_X \circ k^* = z_X^*$ and $z_X^* \circ k^* = z_X^*$. Thus $z_X \circ k^* \circ k^* = z_X \circ 1_X$, so by 1.4(b) it follows that $k^* \circ k^* = 1_{E_{ec}(X)}$. Similarly $k^* \circ k^* = 1_{E_{ec}(X)}$ so $k^*$ is the homeomorphism $h$ required in (c). Hence (c) holds.

To prove (e) note that $\mathcal{B}(X) = \{A \in \mathcal{Z}(X)^*: A$ is complemented in $\mathcal{Z}(X)^*\}$; i.e. $\mathcal{B}(X)$ is characterized (as a subset of $\mathcal{Z}(X)^*$ in lattice-theoretic terms. Thus (e) follows from (d) and 1.4.

We now compare $QF(X)$ and $E_{ec}(X)$ for the compact space $X$.

3.9. Theorem. If $X$ is compact then $(QF(E_{ec}(X)), z_X \circ \varphi_{E_{ec}(X)})$ and $(QF(X), \varphi_X)$ are equivalent (as defined preceding 2.9) covers of $X$.

Proof. By 2.14 and 3.8 there are covering maps $\varphi_{E_{ec}(X)}: QF(E_{ec}(X)) \to X$, $\varphi_X: QF(X) \to X$, and $z_X : E_{ec}(X) \to X$. Let $g = z_X \circ \varphi_{E_{ec}(X)}$, then $g$ is a covering map from $QF(E_{ec}(X))$ onto $X$. Because $(QF(X), \varphi_X)$ is the minimal quasi-F-cover of $X$, there exists a covering map $h: QF(E_{ec}(X)) \to QF(X)$ such that $\varphi_X \circ h = z_X \circ \varphi_{E_{ec}(X)}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.pdf}
\caption{Fig. 2}
\end{figure}

(see Fig. 2). By 3.5 $QF(X)$ is a cloz-space, so by the minimality of $E_{ec}(X)$ there exists a covering map $f: QF(X) \to E_{ec}(X)$ such that $z_X \circ f = \varphi_X$. Thus $z_X \circ \varphi_{E_{ec}(X)} = z_X \circ f \circ h$, so by 1.4(b) it follows that $\varphi_{E_{ec}(X)} = f \circ h$. By 2.14, and the fact that $(QF(X), f)$ is a quasi-F-cover of $E_{ec}(X)$, there exists a covering map $c: QF(X) \to E_{ec}(X)$ such that $\varphi_{E_{ec}(X)} \circ c = f$. Thus $\varphi_{E_{ec}(X)} \circ c \circ h = f \circ h$. 


= \varphi_{E_{cc}(X)} \circ \text{Id}_{QF(X)}$, so as above \( c \circ h = \text{Id}_{QF(X)} \). It follows that \( h \) is a homeomorphism. ■

3.10. Remark. (a) It is evident that 3.9 can be generalized in the following way. Let \( \mathcal{P} \) be a topological property for which each compact space \( X \) possesses a "minimal" \( \mathcal{P} \)-cover \( (\mathcal{P}(X), \varphi_X) \). (In other words, \( \mathcal{P}X \) has \( \mathcal{P} \), \( \varphi_X : \mathcal{P}(X) \to X \) is a covering map, and if \( (Y, f) \) is another \( \mathcal{P} \)-cover of \( X \) then there is a covering map \( g : \mathcal{P}(X) \to Y \) for which \( \varphi_X \circ g = f \). Then if each quasi-\( F \)-space has \( \mathcal{P} \), it follows that \( QF(\mathcal{P}(X)) = QF(X) \).

(b) Theorem 3.9 can be proved more directly as follows. As \( z \)-irreducible maps are closed under composition, by 3.8(d) \( z_X \circ \varphi_{E_{cc}(X)} \) is a \( z \)-irreducible map from \( QF(E_{cc}(X)) \) onto \( X \). As \( (QF(X), \varphi) \) is the projective cover of \( X \) in the category of compact spaces and \( z \)-irreducible maps (see 2.14) there is a covering map \( c : QF(X) \to QF(E_{cc}(X)) \) for which \( \varphi_X = z_X \circ \varphi_{E_{cc}(X)} \circ c \). Theorem 3.9 follows quickly.

It follows quickly from 3.8 that the connectedness of \( QF(X) \) and of \( E_{cc}(X) \) are related. Specifically, we have the following. (The equivalence of (a) and (c) was communicated to us by A. Hager and L. Robertson.)

3.11. Proposition. The following are equivalent for the compact space \( X \).

(a) \( QF(X) \) is connected.
(b) \( E_{cc}(X) \) is connected.
(c) Each dense cozero-set of \( X \) is connected.
(d) \( \mathcal{P}(X) = \{ \emptyset, X \} \).

Proof. Since \( QF(X) \) and \( E_{cc}(X) \) are cloz-spaces (see 3.5 and 3.8), the equivalence of 3.4(a) and (d) immediately implies the equivalence of (a), (b), and (d) above.

(d) \( \Rightarrow \) (c). If (c) fails, let \( C \) be a disconnected dense cozero-set of \( X \). If \( A \) and \( B \) are non-empty complements of \( C \), it follows from 1.1 of [BH] that \( A \) and \( B \) are cozero-sets of \( X \), and hence (as \( C \) is dense in \( X \)) members of \( cc(X) \). Hence \( \mathcal{P}(X) \) contains the four elements \( \emptyset, X, cl_XA, \) and \( cl_XB \), so (d) fails.

(c) \( \Rightarrow \) (d). If (d) fails, there exists a complemented cozero-set \( C \) of \( X \) such that \( \emptyset \neq cl_XC \neq X \). If \( D \) is a cozero-complement of \( C \), then \( \emptyset \neq D \neq X \), and \( C \cup D \) is a disconnected dense cozero-set of \( X \). Hence (c) fails. ■

3.12. Corollary. If \( E_{cc}(X) \) is connected then \( X = E_{cc}(X) \); the converse fails.

Proof. If \( E_{cc}(X) \) is connected, then \( \mathcal{P}(X) = \{ \emptyset, X \} \) by 3.11. It follows from 2.9 that \( z_X \) is a homeomorphism, so \( X = E_{cc}(X) \). The failure of the converse is witnessed by any compact extremally disconnected space \( X \) containing more than one point. ■

We now give an example to illustrate that \( E_{cc}(X) \) and \( QF(X) \) can both be connected, but fail to be homeomorphic.
3.13. Example. Let $T$ denote an infinite compact connected set in which every zero-set is a regular closed set. (Eg. let $T = \beta R^n \setminus R^n$ for $n > 1$, or $\beta([0, +\infty)) \setminus [0, +\infty)$; see 1.62, 4.21, and Chapter 9 of [Wa].) Let $\Sigma$ denote the free union of countably many copies $T_n$ of $T$, and let $\pi_n$ denote a homeomorphism of $T_n$ onto $T$ for each $n \in \mathbb{N}$. Choose, for each $n \in \mathbb{N}$, $p_n \in T_n$ and $q_n \in T_{n+1}$ in such a way that all the pairs $(\pi_n(p_n), \pi_{n+1}(q_n))$ are distinct in $T$. Let $S$ denote the quotient space of $\Sigma$ obtained by identifying $p_n$ in $T_n$ with $q_n$ in $T_{n+1}$ for each $n \in \mathbb{N}$. Then $S$ is a $\tau S$-connected space, as is its one-point compactification $\tau S$. Clearly $S$ is the only proper dense cozero-set of $\tau S$. Hence $S$ is connected, $\tau S$ is a cloz-space. But $S$ is not $C^*$-embedded in $\tau S$, so $\tau S$ is not a quasi-$F$-space. Indeed, it is not hard to see that $QF(\tau S) = \beta S$. In summary, $\tau S = E_{\text{cc}}(\tau S)$ and $QF(\tau S)$ are connected but not homeomorphic. 

Recall that a quasi-$F$-space that contains a proper dense cozero-set is called a proper quasi-$F$-space. It is shown in 5.11 of [DHH] that the product of two infinite proper quasi-$F$-spaces can never be a quasi-$F$-space. We will now show that this need not be the case if “cloz” is substituted for “quasi-$F$” in this latter assertion.

3.14. Example. We exhibit a cloz-space with a proper dense cozero-set whose square is a cloz-space. The space $\tau S$ of 3.12 is a cloz-space, but is not a quasi-$F$-space. Note first that $\tau S \times \tau S$ has no nontrivial pair of complementary cozero-sets. For if $\{C_1, C_2\}$ were such a pair, then $C_1 \cap (S \times S)$ and $C_2 \cap (S \times S)$ are complementary cozero-sets of $S \times S$. But by 5.10 of [DHH] $S \times S$ has no proper dense cozero-sets. Since $S \times S$ is connected, this implies that one of $C_1 \cap (S \times S)$ and $C_2 \cap (S \times S)$ is empty. If $C_2 \cap (S \times S) = \emptyset$ it follows that $C_1 = \emptyset$ and $C_2 = \tau S \times \tau S$.

Now let $X = \tau S \times \{0, 1\}$ denote the free union of two copies of $\tau S$. Then $X \times X$ is the free union of four copies of $\tau S \times \tau S$ and is a cloz-space. Also, it contains the nontrivial pair $C_1 = \tau S \times \{0\}$ and $C_2 = \tau S \times \{1\}$ of complementary cozero-sets, and is the square of a cloz-space with a proper dense cozero-set. 

Finally, we go on to determine whether or not $QF(X)$ and $E_{\text{cc}}(X)$ are homeomorphic, although we do not settle this question completely.

3.15. Theorem. Let $X$ be a compact space. Then:

(a) If $X$ contains a dense totally disconnected cozero-set then $E_{\text{cc}}(X) = QF(X)$ and $QF(X)$ is zero-dimensional.

(b) If $X$ is zero-dimensional then $E_{\text{cc}}(X) = QF(X)$.

(c) The following are equivalent:

(i) $QF(X)$ is zero-dimensional.

(ii) $E_{\text{cc}}(X)$ is zero-dimensional.

(iii) $\mathcal{G}(X)$ is a base for the closed subsets of $X$. 
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Proof. (c): (iii)⇒(ii). Suppose that $\mathcal{G}(X)$ is a base for the closed subsets of $X$. It follows from 2.10 that $E_{\text{ce}}(X) = \mathcal{L}(\mathcal{G}(X))$. As remarked prior to 2.11, since $\mathcal{G}(X)$ is a subalgebra of $\mathcal{B}(X)$, $\mathcal{L}(\mathcal{G}(X))$ is the Stone space of the Boolean algebra $\mathcal{G}(X)$. Hence $E_{\text{ce}}(X)$ is zero-dimensional.

(ii)⇒(i). Assume that $E_{\text{ce}}(X)$ is zero-dimensional, and suppose that $C$ were a dense cozero-set of $E_{\text{ce}}(X)$. Then $C$ is Lindelöf, zero-dimensional, and 2-embedded in $E_{\text{ce}}(X)$. But a dense, zero-dimensional Lindelöf subspace of a compact space is 2-embedded in the space iff it is $C^*$-embedded in the space (this follows from 16.16 of [GJ] and Urysohn’s extension theorem (1.17 of [GJ])). Thus $C$ is $C^*$-embedded in $E_{\text{ce}}(X)$, and it follows that $E_{\text{ce}}(X)$ is a quasi-$F$-space. The minimality of $QF(X)$ implies that there is a covering map $g$: $E_{\text{ce}}(X) \to QF(X)$ such that $\varphi_{\lambda} \circ g = z_{\lambda}$. As noted in the proof of 3.9, there is a covering map $f$: $QF(X) \to E_{\text{ce}}(X)$ such that $z_{\lambda} \circ f = \varphi_{\lambda}$. Hence $\varphi_{\lambda} \circ g \circ f = z_{\lambda} \circ f = \varphi_{\lambda} \circ 1_{QF(X)}$. Then $\varphi_{\lambda} \circ g \circ f = \varphi_{\lambda} \circ g = 1_{E_{\text{ce}}(X)}$. Two applications of 1.4(b) yield that $f$ and $g$ are homeomorphisms, and (i) follows.

(i)⇒(iii). Suppose that $QF(X)$ is zero-dimensional. It suffices to prove that if $A$ is closed in $X$ and $p \in X \setminus A$, there exists $G \in \mathcal{G}(X)$ such that $A \subseteq G$ and $p \in X \setminus G$. By the compactness and complete regularity of $X$ there exist disjoint zero-sets $Z_1$ and $Z_2$ of $X$ such that $A \subseteq \text{int}_X Z_1$ and $p \in \text{int}_X Z_2$. As $\varphi_{\lambda}^{-1}[Z_1]$ and $\varphi_{\lambda}^{-1}[Z_2]$ are disjoint compact subsets of the compact zero-dimensional space $QF(X)$, there exists a clopen subset $B$ of $QF(X)$ such that $\varphi_{\lambda}^{-1}[Z_1] \subseteq B$ and $\varphi_{\lambda}^{-1}[Z_2] \subseteq QF(X) \setminus B$. Now $B \in \mathcal{G}(QF(X))$ and $\varphi_{\lambda}$ is $Z^*$-irreducible so by 2.14 $\varphi_{\lambda}[B]$ and $\varphi_{\lambda}[QF(X) \setminus B]$ are complementary (in $\mathcal{B}(X)^*$ and hence in $\mathcal{G}(X)$) members of $\mathcal{B}(X)^*$, and hence both belong to $\mathcal{G}(X)$. Thus $A \subseteq \text{int}_X \varphi_{\lambda}[B]$, $p \in \varphi_{\lambda}[QF(X) \setminus B]$, and $\text{int}_X \varphi_{\lambda}[B] \cap \text{int}_X \varphi_{\lambda}[QF(X) \setminus B] = \emptyset$. Thus $\varphi_{\lambda}[B]$ is the required $G$ and so (iii) holds.

(a) Let $U$ be a dense cozero-set of $X$ that is totally disconnected. Then $U$ is locally compact and Lindelöf, and hence is zero-dimensional (see 16.17 of [GJ]).

We claim that $\mathcal{G}(X)$ is a base for the closed subsets of $X$. To see this, suppose $A$ is closed in $X$ and $p \in X \setminus A$. By complete regularity and compactness, there exist disjoint zero-sets $Z_0$ and $Z_1$ of $X$ such that $A \subseteq \text{int}_X Z_0$ and $p \in \text{int}_X Z_1$. By 16.16 of [GJ] there is a clopen set $B$ of $U$ such that $Z_0 \cap U \subseteq B$ and $Z_1 \cap U \subseteq U \setminus B$. By 3.7(b) of [HVW] there exists $D \in \mathcal{B}(X)^*$ such that $D \cap U = B$. It is routine to verify that $A \subseteq D$ and $p \in X \setminus D$.

Our claim follows.

It follows that $E_{\text{ce}}(X)$ is zero-dimensional from (c), and the proof that (ii)⇒(i) in (c) proves that $E_{\text{ce}}(X)$ and $QF(X)$ are homeomorphic; hence $QF(X)$ is zero-dimensional. Obviously, (b) is a special case of (a). ■

3.16. Questions. (a) We do not have a characterization of those compact spaces $X$ for which $(QF(X), \varphi_{\lambda})$ and $(E_{\text{ce}}(X), z_{\lambda})$ are equivalent covers of $X$, although 3.15 provides us with partial answers.
(b) In [HVW] a theory of quasi-$F$-covers of Tikhonov spaces is developed, and the relation between $QF(X)$ and $QF(\beta X)$ is explored. The analogous theory for $E_e(X)$ is as yet undeveloped.

(c) A more important problem is the following: find a convergence notion within the class of lattice-ordered rings or vector lattices with respect to which $C(E_e(X))$ is the "completion" of $C(X)$ in a way similar to which $C(QF(X))$ is the completion of $C(X)$ with respect to order-convergence; see [DHH] for details.

§ 4. Wallman compactifications versus Wallman covers

In § 2 we have seen that a variety of covers of a given compact space are of Wallman type; the absolute (more generally, every zero-dimensional cover) and the quasi-$F$-cover are Wallman covers. In this section we consider the question of whether every cover of a compact space is a Wallman cover. By exploiting the relationship between Wallman compactifications and Wallman covers, we can use the example, due to Ul'yanov and Šapiro, of a non-Wallman compactification of an uncountable discrete space to produce a cover of the one-point compactification of an uncountable discrete space that is not a Wallman cover. We use similar arguments to show that whether all covers of the one-point compactification of $N$ are Wallman covers is independent of the usual axioms of set theory (ZFC).

We begin by developing the relationship between the lattice of compactifications of a locally compact extremally disconnected space $X$ and the lattice of covers of the one-point compactification of $X$. Later we consider the special case in which $X$ is discrete. Throughout this section $\gamma X$ will denote the one-point compactification of the locally compact non-compact space $X$, and $\infty$ will denote its "point at infinity"; i.e. $\gamma X \setminus X = \{ \infty \}$. If $\alpha X$ is another compactification of $X$, then $g_\alpha$ will denote the unique continuous surjection from $\alpha X$ onto $\gamma X$ for which $g_\alpha(x) = x$ for all $x \in X$.

Recall that compactifications $\alpha X$ and $\delta X$ of the space $X$ are said to be equivalent if there is a homeomorphism $h$: $\alpha X \to \delta X$ such that $h(x) = x$ for each $x \in X$. As is usual, we identify equivalent compactifications. Let $\mathcal{H}(X)$ denote the set of all compactifications of $X$ (up to equivalence). If $\alpha X$, $\delta X \in \mathcal{H}(X)$ then we say $\alpha X \leq \delta X$ if there is a continuous surjection $f$: $\delta X \to \alpha X$ such that $f(x) = x$ for each $x \in X$. It is well-known that $(\mathcal{H}(X), \leq)$ is a complete upper semilattice, and is a complete lattice iff $X$ is locally compact (see 4.2(a) and 4.3(e) of [PW], for example). The largest member of $\mathcal{H}(X)$ is the Stone-Čech compactification $\beta X$; if $X$ is locally compact, the smallest member is $\gamma X$.

4.1. Lemma. Let $X$ be locally compact. Then:

(a) $(\mathcal{H}(X), \leq)$ is a sublattice of the lattice $(\mathcal{C}(\gamma X), \leq)$ of all covers of $\gamma X$.
(See the remarks following the proof of 2.8.)

(b) $(\mathcal{H}(X), \leq) = (\mathcal{C}(\gamma X), \leq)$ iff $X$ is extremally disconnected.
§ 4. Wallman compactifications versus Wallman covers

Proof. (a) Let \( \alpha X \in \mathcal{X}(X) \). The fact that \( g_\alpha(x) = x \) for each \( x \in X \), and the fact that \( X \) is dense in \( \alpha X \), immediately imply that \( g_\alpha \) is irreducible. Thus \( (\alpha X, g_\alpha) \in \mathcal{C}(\gamma X) \).

(b) Assume that \( X \) is extremally disconnected and suppose \( (K, f) \in \mathcal{C}(\gamma X) \). Then as \( f \) is irreducible, \( f|f^{-1}[X] \) is a perfect irreducible continuous surjection onto the extremally disconnected space \( X \) and hence is a homeomorphism (see 1.5 of [Wo]). As \( X \) is dense in \( \gamma X \) and \( f \) is irreducible, it follows from 1.4(a) that \( f^{-1}[X] \) is dense in \( K \). Thus \( (K, f) \) is equivalent (as a cover of \( \gamma X \)) to some compactification of \( X \) and hence (as equivalent covers are identified) \( K \in \mathcal{X}(X) \).

Now assume \( X \) is locally compact but not extremally disconnected. The absolute \( (E(\gamma X), k_{\gamma X}) \) of \( \gamma X \) is a cover of \( \gamma X \) as was noted in the remarks following 2.13, and \( E(\gamma X) \) is extremally disconnected. However, no compactification of \( X \) can be extremally disconnected as dense subspaces of extremally disconnected spaces are extremally disconnected (see 6M of [GJ]). Thus \( (E(\gamma X), k_{\gamma X}) \) is not equivalent (as a cover of \( \gamma X \)) to any cover of \( \gamma X \) of the form \( (\alpha X, g_\alpha) \), where \( \alpha X \) is a compactification of \( X \); i.e. \( (E(\gamma X), k_{\gamma X}) \in \mathcal{C}(X) \setminus \mathcal{X}(X) \).

The following properties of Wallman compactifications are well known (see, for example 19K (4), (5) of [Wi]) and easily proved.

4.2. Lemma. Let \( \mathcal{C} \) be a Wallman base for a space \( X \), and let \( C, D \in \mathcal{C} \). Then:

(a) \( \text{cl}_{w_{\mathcal{C}}(x)} C = C^* \) (see 1.1(d) for notation).

(b) \( \text{cl}_{w_{\mathcal{C}}(x)}(C \cap D) = \text{cl}_{w_{\mathcal{C}}(x)} C \cap \text{cl}_{w_{\mathcal{C}}(x)} D \).

We will need the following.

4.3. Lemma. Let \( \mathcal{A} \) be a Wallman sublattice of \( \mathcal{R}(\gamma X) \). Suppose \( A \in \mathcal{A} \), \( B \in \mathcal{A} \), and \( B \) is a clopen subset of \( X \). Then \( \text{cl}_{\gamma X}(A \setminus B) \in \mathcal{A} \).

Proof. If \( x \in \gamma X \setminus B \), since \( \mathcal{A} \) is \( T_1 \) with respect to \( X \) there exists \( C(x) \in \mathcal{A} \) such that \( x \in \text{int}_{\gamma X} C(x) \) and \( C(x) \setminus B = \emptyset \). As \( B \) is clopen, it follows that \( C(x) \setminus B = \emptyset \). Since \( \gamma X \setminus B \) is compact (as \( B \) is open), there exists a finite subset \( F \) of \( \gamma X \setminus B \) such that \( \gamma X \setminus B \) is \( \bigcup \{ C(x) : x \in F \} \), which belongs to \( \mathcal{A} \) as \( \mathcal{A} \) is a sublattice of \( \mathcal{R}(\gamma X) \) and hence is closed under finite unions. Evidently, \( A \setminus B = (\gamma X \setminus B) \cap A = (\gamma X \setminus B) \cap A \) (since \( \gamma X \setminus B \) is clopen).

4.4 Theorem. Let \( X \) be a locally compact extremally disconnected space, and let \( \alpha X \) be a compactification of \( X \). Then \( (\alpha X, g_\alpha) \) is (equivalent to) a Wallman cover of \( \gamma X \) if \( \alpha X \) is a regular Wallman compactification of \( X \) (see 1.1(f)).

Proof. Let \( \mathcal{C} \) be a Wallman base for \( X \) consisting of regular closed subsets of \( X \), and denote \( w_{\mathcal{C}}(x) \) by \( \alpha X \). We show that \( (\alpha X, g_\alpha) \) is a Wallman cover of \( \gamma X \). Since \( X \) is extremally disconnected, \( \mathcal{C} \) consists of clopen subsets of \( X \). Since \( \mathcal{C} \) is a ring of sets, it follows that \( \mathcal{C} \) is a sublattice of \( \mathcal{R}(X) \). Since \( X \) is dense in \( \alpha X \), the map \( A \rightarrow \text{cl}_{\alpha X} A \) is a Boolean algebra (and hence lattice) isomorphism from \( \mathcal{R}(X) \) onto \( \mathcal{R}(\alpha X) \); hence \( \{ \text{cl}_{\alpha X} C : C \in \mathcal{C} \} \) is a sublattice \( \mathcal{A} \) of \( \mathcal{R}(\alpha X) \) that is lattice-isomorphic to \( \mathcal{C} \).
We now show that \( \mathcal{A} \) is a Wallman sub lattice of \( \mathcal{R}(\alpha X) \). First, suppose that \( \delta \in \alpha X \setminus \operatorname{clo}_\alpha C \) for some \( C \in \mathcal{C} \). If \( \delta \in \mathcal{C} \), then \( \delta \notin \mathcal{C} \) so as \( \mathcal{C} \) is a Wallman base there exists \( D \in \mathcal{C} \) such that \( \delta \in D \) and \( C \cap D = \emptyset \). As \( D \) is open in \( X \), it follows that \( \delta \in \operatorname{int}_\alpha C \setminus \operatorname{clo}_\alpha D \). As \( A \to \operatorname{clo}_\alpha A \) is a lattice isomorphism from \( \mathcal{R}(X) \) onto \( \mathcal{R}(\alpha X) \), and as \( C \cap D = C \cap D \) in \( \mathcal{R}(X) \) since \( C \) and \( D \) are clopen subsets of \( X \), it follows that \( \operatorname{clo}_\alpha C \cap \operatorname{clo}_\alpha D = \emptyset \) (in \( \mathcal{R}(\alpha X) \)). Thus \( \mathcal{A} \) is \( T_1 \) with respect to \( \alpha X \).

Note that the definition of a normal sublattice of \( \mathcal{R}(X) \) is purely lattice-theoretic (since \( C \cup D = C \cup D \) for \( C, D \in \mathcal{C}(X) \); see 2.1(b)). As \( \mathcal{C} \) is a normal ring of sets (see 1.1(b)) and a sublattice of \( \mathcal{R}(X) \) consisting of clopen sets, it follows that \( C \cap D = C \cap D \) for \( C, D \in \mathcal{C} \) and hence that \( \mathcal{C} \) is a normal sublattice of \( \mathcal{R}(X) \) (see 2.1(b)). As \( \mathcal{A} \) is lattice-isomorphic to \( \mathcal{C} \), it follows that \( \mathcal{A} \) is a normal sublattice of \( \mathcal{R}(\alpha X) \). Hence \( \mathcal{A} \) is a Wallman sub lattice of \( \mathcal{R}(\alpha X) \).

As \( (\alpha X, g_\alpha) \) is a cover of \( X \), the sublattice \( \mathcal{A}(\gamma X, g_\alpha) \) (see 1.5(a)) is, by 2.2 and 2.3, a Wallman sublattice of \( \mathcal{R}(\gamma X) \). Denote it by \( \mathcal{A}^* \). Then \( (\mathcal{L}(\mathcal{A}^*), \gamma X, \psi_{\mathcal{A}^*}) \) (as described in 2.6) is a Wallman cover of \( \gamma X \). Explicitly, \( \mathcal{A}^* = \{ g_\alpha \mid \operatorname{clo}_\alpha C : C \in \mathcal{C} \} \).

Note that \( \mathcal{A} = \{ C^* : C \in \mathcal{C} \} \) by 4.2(a), and hence by definition of \( w_\alpha(X) \) is a base for the closed subsets of \( \alpha X \). This in turn implies that \( \mathcal{A}^* \) is a base for the closed subsets of \( \gamma X \); for if \( x \in \gamma X \setminus K \) and \( K \) is closed in \( \gamma X \), \( g_\alpha[K] \) and \( g_\alpha^{-1}(x) \) are disjoint closed subsets of \( \alpha X \). Let \( G \) be a compact subset of \( \alpha X \) such that \( g_\alpha^{-1}(x) \subseteq \operatorname{int}_\alpha G \subseteq G \subseteq \alpha X \setminus g_\alpha^{-1}(x) \). As \( \mathcal{A} \) is a base for the closed subsets of \( \alpha X \), there exists, for each \( y \in g_\alpha^{-1}(x) \), \( A(\gamma x) \in \mathcal{A} \) such that \( G \subseteq A(\gamma x) \subseteq \alpha X \setminus \{ y \} \). By compactness there exist \( y_1, \ldots, y_n \in g_\alpha^{-1}(x) \) such that \( G \subseteq \bigcap_{i=1}^n A(y_i) \subseteq \alpha X \setminus \{ y \} \). Thus \( g_\alpha^{-1}(x) \subseteq \operatorname{int}_\alpha G \subseteq \bigcap_{i=1}^n A(y_i) = A \in \mathcal{A} \), and \( A \cap g_\alpha^{-1}(x) = \emptyset \). Thus \( K \subseteq g_\alpha[A] \subseteq \gamma X \setminus \{ x \} \), and \( g_\alpha[A] \in \mathcal{A}^* \). Thus \( \mathcal{A}^* \) is a closed base for \( \gamma X \) as claimed.

It thus follows from 2.10 that \( \mathcal{L}(\mathcal{A}^*), \gamma X \) is equivalent (as a cover of \( \gamma X \)) to \( \mathcal{L}(\mathcal{A}^*) \), and that we can regard the map \( \psi_{\mathcal{A}^*} \) as taking each point of \( \mathcal{L}(\mathcal{A}^*) = \) — each ultrafilter on \( \mathcal{A}^* \) — to the unique point of \( \gamma X \) to which it converges.

We now show that \( (\alpha X, g_\alpha) \) and \( (\mathcal{L}(\mathcal{A}^*), \psi_{\mathcal{A}^*}) \) are equivalent covers of \( \gamma X \). Let \( \delta \) be a point of \( \mathcal{L}(\mathcal{A}^*) \), i.e. an ultrafilter on \( \mathcal{A}^* \). As \( C \to \operatorname{clo}_\alpha C \) is an isomorphism from \( \mathcal{C} \) onto \( \mathcal{A} \), and \( A \to g_\alpha[A] \) is (by 1.4) a lattice isomorphism from \( \mathcal{A} \) onto \( \mathcal{A}^* \), it follows that \( \{ C \in \mathcal{C} : g_\alpha[C] \in \delta \} \) is an ultrafilter \( \delta' \) on \( \mathcal{C} \), i.e. a point of \( w_\alpha(X) = \alpha X \) Define \( h : \mathcal{L}(\mathcal{A}^*) \to \alpha X \) by \( h(\delta) = \delta' \) for each \( \delta \in \mathcal{L}(\mathcal{A}^*) \). It is tedious but routine to prove that \( h \) is a homeomorphism satisfying \( g_\alpha h = \psi_{\mathcal{A}^*} \). Thus \( (\alpha X, g_\alpha) \) is equivalent to \( \mathcal{A} \) a Wallman cover of \( \gamma X \).

Conversely, suppose that \( \mathcal{A} \) is a Wallman sub lattice of \( \mathcal{R}(\gamma X) \). In 4.1(b) we saw that (up to equivalence of covers) the associated Wallman cover \( (\mathcal{L}(\mathcal{A}, \gamma X), \psi_{\mathcal{A}}) \) is equivalent to a compactification \( \alpha X \) of \( X \), together with the map \( g_\alpha \).

Thus we regard \( \psi_{\mathcal{A}}[X] \) as being \( X \), and \( \mathcal{L}(\mathcal{A}, X) \) as being \( \alpha X \).

As noted above, \( \{ A \cap X : A \in \mathcal{A} \} = \mathcal{A}_0 \) will be a sublattice of \( \mathcal{R}(X) \) that is lattice-isomorphic to \( \mathcal{A} \). Hence (as noted above) as \( X \) is extremely disconnected,
§ 4. Wallman compactifications versus Wallman covers

each member of $\mathcal{A}_0$ is clopen in $X$. Let $\mathcal{F} = \{F \subseteq X: F$ is clopen in $X$, and either $F$ or $X \setminus F$ is compact}. It is easily verified that $\mathcal{F}$ is a subalgebra of $\mathcal{B}(X)$. Let $\mathcal{A}_1$ be the sublattice of $\mathcal{B}(X)$ generated by $\mathcal{A}_0 \cup \mathcal{F}$. It is easily verified that

$$\mathcal{A}_1 = \{F \in \mathcal{B}(X): F = (A \setminus G) \cup H \text{ for some } A \in \mathcal{A}_0 \text{ and compact open subsets } H \text{ and } G \text{ of } X\}.$$

We claim that $\mathcal{A}_1$ is a Wallman base for $X$. As $X$ is locally compact and zero-dimensional, the set of complements of compact open sets of $X$ forms a base for the closed subsets of $X$, and so $\mathcal{A}_1$ is a base for the closed sets of $X$. As $\mathcal{A}_1$ is a sublattice of $\mathcal{B}(X)$ and $A \cap B = A \cap B$ for each $A, B \in \mathcal{A}_1$, since $X$ is extremely disconnected, it follows that $\mathcal{A}_1$ is a ring of sets. Now suppose $S \in \mathcal{A}_1$ and $x \in X \setminus S$. Then $S = (A \setminus G) \cup H$ for some $A \in \mathcal{A}_0$ and compact open subsets $G$ and $H$ of $X$. If $x \notin A$, since $\mathcal{A}_1$ is a Wallman sublattice of $\mathcal{B}(\gamma X)$ there exists $D \in \mathcal{A}$ such that $x \in \text{int}_X D$ and $(\text{cl}_X A) \setminus D = \emptyset$ (see 2.1(a)). Let $B = D \cap X$. Then $B \in \mathcal{A}_1$, $x \in B \cap H$, and $(B \setminus H) \cap S = \emptyset$ since $X$ is extremely disconnected. If $x \in A$, then $x \in (\text{cl}_X H) \cap \mathcal{A}_1$ and $(G \setminus H) \cap S = \emptyset$. It follows that $\mathcal{A}_1$ is a ring of sets that is $T_1$ (see 1.1(a)).

Finally, we outline the tedious computation needed to show that $\mathcal{A}_1$ is a normal ring. Suppose that $S_1, S_2 \in \mathcal{A}_1$ and $S_1 \cap S_2 = \emptyset$. Then $S_1 = (A_1 \setminus G_1) \cup H_1$ and $S_2 = (A_2 \setminus G_2) \cup H_2$ where $A_1, A_2 \in \mathcal{A}_0$ and $G_1, G_2, H_1, H_2$ are compact open subsets of $X$. As $S_1 \cap S_2 = \emptyset$, it follows that:

1. $H_1 \cap H_2 = \emptyset$,
2. $A_1 \cap A_2 \subseteq G_1 \cup G_2$,
3. $A_1 \cap H_2 \subseteq G_1$,
4. $A_2 \cap H_1 \subseteq G_2$.

By (2) $A_1 \cap A_2$ is a compact open subset of $X$, and hence of $\gamma X$, and it quickly follows that $(\text{cl}_X A_1) \setminus (\text{cl}_X A_2) = A_1 \cap A_2$ (infimum taken in $\mathcal{A}$). Thus $A_1 \cap A_2 \in \mathcal{A}$, and it follows from 4.3 that $A_1 \setminus A_2 \in \mathcal{A}$. As $\mathcal{A}$ is a normal sublattice of $\mathcal{B}(\gamma X)$, there exist $C, D \in \mathcal{A}$ such that $(A_1 \setminus A_2) \cap C = A_2 \cap D = \emptyset$ and $C \cup D = \gamma X$. Let $C = X \cap C'$ and $D = H \cap D'$.

If we set

$$T_1 = \left[\left((C \setminus (A_1 \cap A_2)) \cup G_1\right) \setminus H_1\right] \cup H_2,$$

$$T_2 = \left[\left(D \cup G_2\right) \setminus H_2\right] \cup H_1,$$

then it can be verified that $S_1 \cap T_1 = S_2 \cap T_2 = \emptyset$, $T_1 \cup T_2 = X$, and $T_1, T_2 \in \mathcal{A}$. Thus $\mathcal{A}_1$ is a normal ring of closed subsets of $X$, and hence is a Wallman base for $X$.

We now show that the Wallman compactification $w_{\mathcal{A}_1}(X)$ is equivalent (as a compactification of $X$) to $\alpha X = (\mathcal{L}(\mathcal{A}, \gamma X), \psi_\mathcal{A})$.

**Claim 1.** If $\mathcal{F} \in w_{\mathcal{A}_1}(X)$, then $X$ (i.e. if $\mathcal{F}$ is a free ultrafilter on $\mathcal{A}_1$), then:
(a) Each co-compact clopen subset of $X$ is in $\mathcal{F}$, and
(b) each member of $\mathcal{F}$ contains a member of $\mathcal{F}$ of the form $A \cap S$, where $A \in \mathcal{A}_0$ and $S$ is a co-compact clopen subset of $X$.

To verify (a) note that if $G$ is a compact clopen subset of $X$, then $G \notin \mathcal{F}$ as $\bigcap \mathcal{F} = \emptyset$. Thus $(X \setminus G) \cap F \neq \emptyset$ for each $F \in \mathcal{F}$, so $X \setminus G \in \mathcal{F}$ by the maximality of $\mathcal{F}$. To verify (b), note that if $F \in \mathcal{F}$ then $F$ has the form $(A \setminus G) \cup H$, where $A \in \mathcal{A}_0$ and $G, H$ are clopen compact subsets of $X$. It follows that $F \cap (X \setminus H) \in \mathcal{F}$ and $F \cap (X \setminus H) \subseteq A \cap (X \setminus G)$. Our claim follows.

If $F \in w_{\mathcal{A}_1}(X) \setminus X$, define $\mathcal{U}(\mathcal{F})$ as follows:

$$\mathcal{U}(\mathcal{F}) = \{A \in \mathcal{A} : A \cap X \in \mathcal{F}\}.$$

**Claim 2.** $\mathcal{U}(\mathcal{F})$ is an ultrafilter on $\mathcal{A}$ and $\infty \in \bigcap \mathcal{U}(\mathcal{F})$.

Since $\mathcal{F}$ is a filter on $\mathcal{A}_1$ and $(A \wedge B) \cap X = (A \cap X) \cap (B \cap X)$ whenever $A, B \in \mathcal{A}$ (infimum taken in $\mathcal{A}$), it quickly follows that $\mathcal{U}(\mathcal{F})$ is a filter on $\mathcal{A}$. Now suppose $A \in \mathcal{A} \setminus \mathcal{U}(\mathcal{F})$; we will find $C \in \mathcal{U}(\mathcal{F})$ such that $A \wedge C = \emptyset$, thereby verifying that $\mathcal{U}(\mathcal{F})$ is an ultrafilter on $\mathcal{A}$. If $A \subseteq X$ then $\mathcal{A}$ is a clopen subset of $\gamma X$ and by 4.3, $\gamma X \setminus A \in \mathcal{A}$. By Claim 1, $X \setminus A \in \mathcal{F}$, so $\gamma X \setminus A \in \mathcal{U}(\mathcal{F})$ and $(\gamma X \setminus A) \wedge A = \emptyset$. If $A \setminus X \neq \emptyset$, since $A \notin \mathcal{U}(\mathcal{F})$ we know $A \cap X \notin \mathcal{F}$. By the maximality of $\mathcal{F}$ and Claim 2 we can find a co-compact clopen subset $S$ of $X$ and $B \in \mathcal{A}$ such that $(A \cap X) \cap (B \cap X \cap S) = \emptyset$ and $B \cap X \cap S \in \mathcal{F}$. Thus $A \wedge B$ is a compact subset of $X$ contained in $X \setminus S$ and evidently belongs to $\mathcal{A}$. Let $J = \text{cl}_X(B \setminus (A \wedge B))$. By 4.3 $J \in \mathcal{A}$, and evidently $J \cap X$ contains $B \cap X \cap S$. Thus $J \cap X \in \mathcal{F}$, and so $J \in \mathcal{U}(\mathcal{F})$. Obviously $J \wedge A = \emptyset$, so $J$ is the required $C$. Hence $\mathcal{U}(\mathcal{F})$ is an ultrafilter on $\mathcal{A}$ as claimed. Finally, note that if $\infty \notin \bigcap \mathcal{U}(\mathcal{F})$, then there exists $A \in \mathcal{U}(\mathcal{F})$ such that $A \subseteq X$. Thus $A$ is a compact member of $\mathcal{F}$, and $\mathcal{F}$ could not be free. Hence $\infty \in \bigcap \mathcal{U}(\mathcal{F})$.

We now define a map $g$ from $w_{\mathcal{A}_1}(X)$ onto $\mathcal{L}(\mathcal{A}, \gamma X)$. Recall that we are identifying $\psi_{\mathcal{A}_1}[X]$, a dense subset of $\mathcal{L}(\mathcal{A}, \gamma X)$, with its homeomorph $X$.

(1) If $x \in X$, define $g(x)$ to be the unique point in $\psi_{\mathcal{A}_1}(x)$, i.e. to be the point $(x, x)$, where $x$ is the unique ultrafilter on $\mathcal{A}$ for which $x \in \bigcap \mathcal{A}$ (see 2.6), and recall that by 4.1(b) $|\psi_{\mathcal{A}_1}(x)| = 1$.

(2) If $F \in w_{\mathcal{A}_1}(X) \setminus X$, define $g(F)$ to be $(\mathcal{U}(\mathcal{F}), \infty)$ as defined above (note that $(\mathcal{U}(\mathcal{F}), \infty) \in \mathcal{L}(\mathcal{A}, \gamma X)$ as $\infty \in \bigcap \mathcal{U}(\mathcal{F}))$.

We will show that $g$ is a homeomorphism from $w_{\mathcal{A}_1}(X)$ onto $\mathcal{L}(\mathcal{A}, \gamma X)$ that fixes $X$ pointwise, in the sense that for each $x \in X$, $g$ takes $x$ to the point in $\psi_{\mathcal{A}_1}[X]$ corresponding (in our identification of $X$ with $\psi_{\mathcal{A}_1}[X]$) to $x$, namely $(x, x)$ as described above.

Obviously, $g|X$ is one-to-one, and if $x \in X$ and $F \in w_{\mathcal{A}_1}(X) \setminus X$ then $g(x) \neq g(F)$. Hence to show $g$ is one-to-one it suffices to show that if $F$ and $G$ are distinct members of $w_{\mathcal{A}_1}(X)$ then $\mathcal{U}(F) \neq \mathcal{U}(G)$. If $F = G$ then there exist $F \in \mathcal{F}$, $G \in \mathcal{G}$ with $F \cap G = \emptyset$; it follows from Claim 1 that there exists
\[ A_1, A_2 \in \mathcal{A}, \text{ and co-compact clopen subsets } S_1 \text{ and } S_2 \text{ of } X, \text{ such that } A_1 \cap S_1 \in \mathcal{F}, \ A_2 \cap S_2 \in \mathcal{F}, \text{ and } A_1 \cap S_1 \cap A_2 \cap S_2 \in \emptyset. \] Thus \( A_1 \in \mathcal{W}(\mathcal{F}), \ A_2 \in \mathcal{W}(\mathcal{G}), \text{ and } A_1 \wedge A_2 \) is a compact subset of \( X \). If \( A_2 \in \mathcal{W}(\mathcal{F}) \) then \( A_1 \wedge A_2 \in \mathcal{W}(\mathcal{F}) \) and so \( \infty \notin \mathcal{W}(\mathcal{F}), \) which contradicts our earlier findings. Thus \( A_2 \notin \mathcal{W}(\mathcal{F}) \) and \( A_2 \notin \mathcal{W}(\mathcal{G}), \) so \( \mathcal{W}(\mathcal{F}) \neq \mathcal{W}(\mathcal{G}). \) Hence \( g \) is one-to-one.

Next we show that \( g \) is onto. If \( (x, \alpha) \in \psi_{\omega_1}(X) \) then \( g(x) = (\alpha, x) \); hence \( g \) maps \( X \) onto \( \psi_{\omega_1}[X] \). Now suppose \( y \in \mathcal{L}(\mathcal{A}, X) \setminus \psi_{\omega_1}[X] \). Then \( y = (\mathcal{G}, \infty) \) where \( \mathcal{G} \) is an ultrafilter on \( \mathcal{A} \) and \( \infty \in (\setminus \mathcal{G}) \). It is straightforward to show that if we define \( \mathcal{H} \) to be

\[ \{(G,F) \in H: G \in \mathcal{G}, \text{ and } F \text{ and } H \text{ are compact open subsets of } X\} \]

then \( \mathcal{F} \in \omega_{\omega_1}(X) \setminus X \) and \( g(\mathcal{F}) = (\mathcal{G}, \infty) = y \). Thus \( g \) is surjective.

Finally, we show that \( g \) is continuous. First note that \( g|X \) is a homeomorphism from \( X \) onto \( \psi_{\omega_1}[X] \) (which, recall, we have identified with \( X \)). Since \( X \) is locally compact, it is open in \( \omega_{\omega_1}(X) \) and it immediately follows that \( g \) is continuous at each point of \( X \). If \( \mathcal{F} \in \omega_{\omega_1}(X) \setminus X \), let \( (\mathcal{L}(\mathcal{A}) \setminus \mathcal{A}_0^\bullet) \times V \) be a typical basic open set of \( \mathcal{L}(\mathcal{A}) \times \gamma X \) that contains \( g(\mathcal{F}) \) (see 2.4 and the proof of 2.8 for notation). Thus \( (\mathcal{W}(\mathcal{F}), \infty) \in (\mathcal{L}(\mathcal{A}) \setminus \mathcal{A}_0^\bullet) \times V \) so \( A_0 \notin \mathcal{W}(\mathcal{F}) \) and \( \infty \in V \). Thus \( X \setminus V \) is a compact subset of \( X \), and by the maximality of \( \mathcal{W}(\mathcal{F}) \) there exists \( B_1 \in \mathcal{W}(\mathcal{F}) \) such that \( A_0 \cap B_1 = \emptyset \). Since \( \mathcal{A} \) is a normal sublattice of \( \mathcal{R}(\gamma X) \), there exist \( C, D \in \mathcal{A} \) such that \( B_1 \subseteq C, \ A_0 \subseteq D, \ C \cup D = \gamma X, \) and \( A_0 \cap C = B_1 \cap D = \emptyset \). Thus \( D \notin \mathcal{W}(\mathcal{F}) \), and so \( D \cap X \notin \mathcal{F} \). Thus \( \mathcal{F} \in \omega_{\omega_1}(X) \setminus \omega_{\omega_1}(X) \setminus (D \cap X) \). As \( X \setminus V \) is a compact subset of \( X \), \( \omega_{\omega_1}(X) \setminus (X \setminus V) \) is an open subset of \( \omega_{\omega_1}(X) \) that contains \( \mathcal{F} \). Let \( W = \omega_{\omega_1}(X) \setminus (\omega_{\omega_1}(X) \setminus (D \cap X) \cup (X \setminus V)) \); then \( W \) is open in \( \omega_{\omega_1}(X) \) and contains \( \mathcal{F} \). We will show that \( g[W] \subseteq (\mathcal{L}(\mathcal{A}) \setminus \mathcal{A}_0^\bullet) \times V \), thereby verifying the continuity of \( g \) at \( \mathcal{F} \).

If \( x \in W \cap X \), then \( g(x) = (\alpha, x) \), where \( \alpha \) is the unique ultrafilter on \( \mathcal{A} \) that converges to \( x \). Then \( x \in V \). As \( \alpha \in (\setminus \mathcal{F}) \cap X \), it follows that \( D \notin \mathcal{F} \), so \( \alpha \notin \mathcal{F} \). As \( A_0 \subseteq D, \ \alpha \notin \mathcal{A}_0^\bullet \) so \( g(\alpha) = (\alpha, x) \in (\mathcal{L}(\mathcal{A}) \setminus \mathcal{A}_0^\bullet) \times V \). If \( \mathcal{F} \in \omega_{\omega_1}(X) \setminus W \), then \( g(\mathcal{F}) = (\mathcal{W}(\mathcal{G}), \infty) = \{ A \in \mathcal{A}: A \cap X \in \mathcal{G}, \infty \} \). As \( \mathcal{G} \in \mathcal{W}, \ D \cap X \notin \mathcal{F} \) so \( \mathcal{F} \notin \mathcal{W}(\mathcal{F}) \) and hence \( A_0 \notin \mathcal{W}(\mathcal{F}) \) as \( A_0 \subseteq D \). Thus \( \mathcal{W}(\mathcal{F}) \notin (\mathcal{L}(\mathcal{A}) \setminus \mathcal{A}_0^\bullet) \times V \). Since \( \infty \in V \), evidently \( g(\mathcal{F}) \in (\mathcal{L}(\mathcal{A}) \setminus \mathcal{A}_0^\bullet) \times V \). Thus \( g[W] \subseteq (\mathcal{L}(\mathcal{A}) \setminus \mathcal{A}_0^\bullet) \times V \), so \( g \) is a continuous bijection with compact domain and hence a homeomorphism fixing \( X \) pointwise. Thus \( \omega_{\omega_1}(X) \) and \( \mathcal{L}(\mathcal{A}, X) \) are equivalent compactifications of \( X \).

We can now use known results concerning Wallman compactifications of discrete spaces to produce a compact space which has covers that are not Wallman covers. The following result, combining results of U'yanov [UI] and Šapiro [S], is a special case of Corollary 2 of [UI].
4.5. **Theorem.** If $\alpha$ is a cardinal for which $2^\alpha \geq N_2$, then the discrete space of cardinality $\alpha$ has a non-Wallman compactification.

On the other hand, Bandt [Ba] has proved:

4.6. **Theorem.** If $2^{\aleph_0} = N_1$, then every compactification of $N$ is of Wallman type.

Combining 4.4, 4.5, and 4.6, and noting that discrete spaces are locally compact and extremely disconnected, we immediately obtain:

4.7. **Theorem.** (a) If $D$ is an uncountable discrete space, then $\gamma D$ has covers that are not Wallman covers.

(b) The space $\gamma N$ has covers that are not Wallman covers iff the continuum hypothesis fails.

(Of course, every Wallman compactification of a discrete space is a regular Wallman compactification).

We can generalize 4.7 by making use of the following result, which is 1.1 of [MV].

4.8. **Proposition.** Let $X$ be a Tikhonov space each compactification of which is a Wallman compactification. Let $B$ be a closed subset of $X$. If either $X$ is normal or $B$ is a $C$-embedded copy of $N$, then every compactification of $B$ is a Wallman compactification.

Now we generalize 4.7.

4.9. **Theorem.** Let $X$ be a locally compact non-compact extremely disconnected space.

1. Suppose $X$ is not pseudocompact. If every cover of $\gamma X$ is a Wallman cover of $\gamma X$, then the continuum hypothesis holds.

2. Suppose $X$ is normal and contains an uncountable closed discrete subset. Then not all covers of $\gamma X$ are Wallman covers of $\gamma X$.

**Proof.** (1) By 1.21 of [GJ] $X$ contains a $C$-embedded copy $B$ of $N$. By 4.4, every compactification of $X$ is a regular Wallman compactification. By 4.8 every compactification of $B$ is a Wallman compactification. Hence by 4.5 the continuum hypothesis holds.

(2) Let $D$ be an uncountable closed discrete subset of $X$. By 4.4 if all covers of $\gamma X$ were Wallman covers, then all compactifications of $X$ would be Wallman compactifications of $X$. By 4.8 it would follow that every compactification of $B$ is a Wallman compactification, which would contradict 4.5. Hence $\gamma X$ has covers that are not Wallman covers. $\blacksquare$
References


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