**F-Spaces and Substonean Spaces**

**General Topology as a Tool in Functional Analysis**

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**INTRODUCTION**

K. Grove and G. Pedersen\(^8\) define a substonean space to be a locally compact (Hausdorff) space in which disjoint σ-compact open subspaces have disjoint compact closures. It is routine to verify that a locally compact space \(X\) is substonean if and only if every continuous \(f: S \rightarrow K\), where \(K\) is a compact, has a unique continuous extension \(\tilde{f}: Cl_\sigma S \rightarrow K\) whenever \(S\) is a σ-compact open subspace of \(X\). Spaces with the property obtained by deleting "σ-compact" in the above are called stonean spaces and must be compact. If the only requirement is that open subspaces have open closures, such spaces are said to be extremally disconnected. Thus, a space is stonean if and only if it is compact and extremally disconnected.

As in reference 8, we denote the ring of continuous real-valued functions on a (Tychonoff) space \(X\) by \(C(X)\) and the subring of bounded elements of \(C(X)\) by \(C^*(X)\). In reference 7 (chapter 14), a Tychonoff space is called an F-space if whenever \(f \in C(X)\), there is a \(k \in C(X)\) such that \(f = k|f|\); equivalently, \(X\) is an F-space if and only if whenever \(S_1\) and \(S_2\) are disjoint cozero sets, there is a \(g \in C(X)\) such that \(g|\mathbb{S}_1| = 0\) and \(g|\mathbb{S}_2| = 1\). [Recall that a subset \(Z\) of \(X\) is called a zero set (resp., cozero set) if it is the set on which some \(f \in C(X)\) vanishes (resp., fails to vanish).] That is, disjoint cozero sets of \(X\) are completely separated.

In reference 9 (p. 124), the authors state, “Gillman and Henriksen studied substonean spaces under the name of F-spaces and in the category of completely regular spaces.” In addition, they refer to work on measures on F-spaces due to Seever\(^6\) and to work by Choquet.\(^12\) They announce that the main purpose of reference 9 is to give appropriate background on substonean spaces in order to make it possible to solve problems on continuous diagonalization of matrices with entries from \(C(X)\) for a compact space \(X\) (see reference 10). This work, in turn, has considerable overlap with the contents of Gillman and Henriksen.\(^7\)

Infinite discrete spaces are F-spaces without being substonean. However, within the class of compact spaces, it is easy to see that there is no difference between substonean spaces and F-spaces or between Rickart spaces and basically disconnected spaces. Missing definitions are supplied in the next section, which is devoted to exploring the relationships between these classes of spaces in the category of locally compact spaces.
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Grove and Pedersen, though, seem to be unaware of a lot of the literature in general topology concerning F-spaces and basically disconnected spaces since the publication of reference 8 in 1960, except, seemingly, for papers concerned directly with problems in functional analysis. For example, no reference is made to the book by Walker. Hence, our last section is devoted to showing how many of the results in reference 9 can be derived easily from previously known results and how some of them can be improved.

It seems unfortunate to us that mathematics has become a tower of Babel in which specialists in different areas communicate poorly, if at all.

F-SPACES, SUBSTONEAN SPACES, BASICALLY DISCONNECTED SPACES, AND RICKART SPACES

All topological spaces considered will be Tychonoff spaces unless the contrary is stated. In particular, locally compact spaces are assumed to be Hausdorff spaces. Readers are referred to references 8, 15, and 20 for general background.

Let $X$ be locally compact. Let $\alpha X = X$ if $X$ is compact and let $\alpha X = X \cup \{\infty\}$ denote the one-point compactification if $X$ is not compact. Thus, $\infty \notin X$ and $X$ is an open neighborhood of $\infty$ intersects $X$ in a compact set.

A nonempty subset $S$ of a topological space $X$ is called a $P$-set if it is closed and any $G_\delta$-set containing $S$ is a neighborhood of $S$. (Some authors require that $P$-sets be compact, but we will not.) A one-point $P$-set is called a $P$-point. Note that a nonempty subspace $K$ of $X$ is a $P$-set if and only if each $f \subseteq C(X)$ that vanishes on $S$ vanishes on a neighborhood of $S$ or, equivalently, if and only if whenever $S$ is a cozero set of $X$ disjoint from $K$, then $K \cap C_0 S = \emptyset$.

For $f \subseteq C(X)$, let $Z(f) = \{x \in X : f(x) = 0\}$ and $coz f = X \setminus Z(f)$.

The following describes the relationship between substonean spaces and $F$-spaces.

**Theorem 1.** If $X$ is a locally compact space, then the following assertions are equivalent:

(a) $X$ is substonean;
(b) $\alpha X$ is an $F$-space and either $X$ is compact or $\infty$ is a $P$-point of $\alpha X$;
(c) $X$ is an open subspace of a compact $F$-space $Y$ and either $X = Y$ or $(Y \setminus X)$ is a $P$-set of $Y$.

**Proof:** As was noted in the Introduction, these assertions are clearly equivalent if $X$ is compact. Therefore, we may assume that $\alpha X \neq X$.

Assume that (a) holds and suppose $C_1$ and $C_2$ are disjoint cozero sets of $\alpha X$. Then, $C_1$ and $C_2$ are $\sigma$-compact. We consider two cases (see below).

Suppose first that $\alpha \notin C_1 \cup C_2$. Clearly, $C_1$ and $C_2$ are $\sigma$-compact. Because $X$ is substonean, $C_{\alpha \cap \alpha C_1} \cap C_{\alpha \cap \alpha C_2} = \emptyset$ and both closures are compact. Hence, $C_{\alpha \cap \alpha C_1} = C_{\alpha C_1}$, so $C_{\alpha \cap \alpha C_1} \cap C_{\alpha \cap \alpha C_2} = \emptyset$.

If, instead, $\alpha \in C_1 \cup C_2$, we may assume that $\alpha \in C_2 = \{x : f(x) > 0\}$ for some $f \geq 0$ in $C(\alpha X)$. Then, $\delta - f(\alpha) > 0$. Let $C_1 = C_2 \setminus \delta^{-1}(\delta - \alpha)$. Clearly, $C_1$ and $C_2$ are disjoint $\sigma$-compact open subsets of $\alpha X$ whose closures in $\alpha X$ are contained in $X$. Therefore, because $X$ is substonean,

(i) $C_{\alpha \cap \alpha C_1} \cap C_{\alpha \cap \alpha C_2} = \emptyset$. 
Now,
\[ \mathcal{C}_\alpha C_1 \cap \mathcal{C}_\alpha C_2 = [\mathcal{C}_\alpha C_1 \cap \mathcal{C}_\alpha C_2] \cup [\mathcal{C}_\alpha C_1 \cap f^{-1}([b, \infty)). \]

The first of these latter two intersections is empty by (i). The second intersection is empty as well because \( C_1 \subseteq Z(f) \) and \( f(\infty) = \delta > 0 \). Thus, \( \mathcal{C}_\alpha C_1 \cap \mathcal{C}_\alpha C_2 = \varnothing \), so \( \alpha X \) is an F-space because disjoint closed subsets of a compact space are completely separated.

If \( \infty \) was not a P-point of \( \alpha X \), there would be a \( g \in C(\alpha X) \) such that \( \infty \in \mathcal{C}_\alpha (\text{coz } g) \) and \( g(\infty) = 0 \). Then, \( \text{coz } g \) would be a \( \sigma \)-compact open subset of \( X \) whose closure in \( X \) is not compact, which is contrary to the assumption that \( X \) is substonean. Hence, (a) implies (b).

It also is obvious that (b) implies (c).

Finally, if (c) holds, then \( X \) is a proper open subset of a compact F-space \( Y \) such that \( K = Y \setminus X \) is a P-set of \( Y \). If \( |C_1, C_2| \) is a pair of disjoint \( \sigma \)-compact open subsets of \( X \), then \( C_1, C_2 \) are disjoint cozero sets in \( Y \), whence \( \mathcal{C}_\gamma C_1 \cap \mathcal{C}_\gamma C_2 = \varnothing \) because \( Y \) is an F-space. Furthermore, because \( K \) is a P-set of \( Y \), \( K \cap \mathcal{C}_\gamma C_i = \varnothing \) for \( i = 1, 2 \). Therefore, \( \mathcal{C}_\alpha C_2 = \mathcal{C}_\gamma C \) is compact for each \( i \) and it follows that \( X \) is substonean.

This completes the proof of THEOREM 1.

A subspace \( Y \) of a space \( X \) is said to be \( C^* \)-embedded in \( X \) if the map that sends each \( f \in C^*(X) \) onto its restriction to \( Y \) is a surjection.

Suppose \( Y \) is a closed subspace of the substonean space \( X \). If \( Y \) is compact, then it is \( C^* \)-embedded in \( \alpha X \), which by THEOREM 1 is an F-space. As \( C^* \)-embedded subspaces of F-spaces are F-spaces (see reference 8, corollary 14.26), \( Y \) is a compact F-space and hence is substonean. If \( Y \) is not compact, then \( \alpha Y = \mathcal{C}_\alpha Y = Y \cup \{\infty\} \) and, arguing as above, we see that \( \alpha Y \) is an F-space. As \( \infty \) is a P-point of \( \alpha X \), it is a P-point of \( \alpha Y \). Thus, by THEOREM 1, \( Y \) is substonean. This establishes the following corollary, which appears in reference 9 (corollary 1.4).

**Corollary 2.** A closed subspace of a substonean space is substonean.

A space \( X \) in which every two disjoint cozero sets have disjoint closures is called an F'-space. Clearly, every F-space is an F'-space and a normal F'-space is an F-space. In Gillman and Henriksen, there is an example of a (nonnormal) F'-space that was not an F-space, and other such examples will be cited below.

It is worthwhile to recall from Kohls' that

(1) every open subspace of an F'-space is an F'-space.

Another characterization of substonean spaces is given next.

**Theorem 3.** The following assertions about a Tychonoff space \( X \) are equivalent:

(a) \( X \) is substonean;

(b) \( X \) is a locally compact F'-space in which every \( \sigma \)-compact subset has compact closure;

(c) \( X \) is a locally compact F'-space in which every \( \sigma \)-compact open subset has compact closure.

**Proof.** Suppose (a) holds. It is obvious that (b) holds if \( X \) is compact. Suppose \( X \) fails to be compact, let \( \{K_n; n < \omega\} \) be a countable collection of compact subsets of \( X \),
and let \( S = \bigcup \{ K_n : n < \omega \} \). Because \( \omega \in \bigcup K_n \) for each \( n < \omega \), there exists \( f_\omega \in C(\alpha X) \) such that \( f_\omega(\omega) = 0 \) and \( f_\omega[K_n] = \{ 1 \} \). Let \( C = \bigcup \{ \text{coz } f_\omega : n < \omega \} \). Then, \( C \) is a cozero set of \( \alpha X \) and \( \omega \in \bigcup C \). As \( \omega \) is a \( P \)-point of \( \alpha X \), it follows that \( \omega \in \bigcup C \), and so \( \text{Cl}_\alpha C = \text{Cl}_\alpha \omega C \). Hence, \( \text{Cl}_\alpha C \) is compact and, because \( S \subseteq C \), it follows that \( \text{Cl}_\alpha S \) is compact as well. By Theorem 1, \( X \) is an open subspace of the \( F \)-space \( \alpha X \) and, thus, it is an \( F \)-space by (1). Therefore, (b) holds.

It is obvious that (b) implies (c).

Finally, because a \( \sigma \)-compact open subspace of a locally compact space is a cozero set, (c) implies (a).

We now turn to the relationship between \( F \)-spaces and substonean spaces. As noted in the Introduction, infinite discrete spaces are \( F \)-spaces that fail to be substonean. The converse also fails, but this cannot be shown in such a simple way.

It follows from a celebrated theorem of Fine and Gillman that if \( X \) is a compact \( F \)-space, if the cardinality \( |C^*(X)| \) of \( C^*(X) \) is \( c \), and if the continuum hypothesis (CH) holds, then every open subspace of \( X \) is an \( F \)-space. Thus, we have the following proposition from Theorem 1.

**Proposition 4 (CH).** If \( X \) is substonean and \( |C^*(X)| = c \), then \( X \) is an \( F \) space.

Next, we use an example due to A. Dow (2.2.2.3) to show that the conclusion of Proposition 4 does not follow just from the hypothesis that \( X \) is substonean.

**Example 5.** A substonean space that is not an \( F \)-space.

Let \( \omega \) denote the countably infinite discrete space, let \( \beta \omega \) denote its Stone-Čech compactification, and let \( \omega^* = \beta \omega \setminus \omega \). As usual, an ordinal \( \gamma \) is identified with the space of ordinals less than \( \gamma \) with the interval topology.

Let \( X \) denote \( (\omega_2 + 1) \setminus \{ \text{all countable limit ordinals in } (\omega_2 + 1) \} \), let \( K = \beta (X \times \omega^*) \), let \( L \) denote the nonisolated points of \( X \) [=the set of uncountable limit ordinals of \( (\omega_2 + 1) \)], and let \( C \) denote a cozero set of \( \omega^* \) that fails to be closed. Finally, let \( U = K \setminus \text{Cl}_\chi (L \times C) \). It is noted in Dow (2.1.1) that \( X \) is a Lindelöf space and a \( P \)-space and that \( K \) is an \( F \)-space because the \( F \)-space \( X \times \omega \) is \( C^* \)-embedded in \( K \). Now, \( U \), being an open subset of the \( F \)-space \( K \), is an \( F \)-space as noted in (1) above. By Theorem 3, we will know that \( U \) is substonean once we establish the following:

(2) if \( W \) is an open \( \sigma \)-compact subspace of \( U \), then \( \text{Cl}_\chi W \) is compact.

Because it is shown in Dow (2.3.3) that \( U \) is not an \( F \)-space, this will establish what we have claimed.

We will begin by showing

(3) \( \text{Cl}_\chi W \cap (L \times C) = \emptyset \).

To see this, we first note that because \( X \times \omega^* \) is a Lindelöf space, \( K = \beta (X \times \omega^*) \) is a compact zero-dimensional space. Therefore, we may write \( W = \bigcup_{\alpha \in \omega} A_\alpha \), where each \( A_\alpha \) is a clopen subspace of \( K \). Also, for similar reasons, \( C = \bigcup_{\alpha \in \omega} B_\alpha \), where each \( B_\alpha \) is a clopen (compact) subspace of \( \omega^* \). Thus,

\[
\text{Cl}_\chi W = \text{Cl}_\chi (W \cap (X \times \omega^*))
= \text{Cl}_\chi \left( \bigcup_{\alpha \in \omega} (A_\alpha \cap (X \times \omega^*)) \right).
\]
Note that for each \( n, k < \omega \), \( A_n \cap (X \times B_k) \) is clopen in \( K \), as is the image \( \Pi_X[A_n \cap (X \times B_k)] \) of this latter set under the projection map \( \Pi_X: X \times \omega^* \to X \).

Now, \( W \cap (L \times C) = \emptyset \); thus, \( A_n \cap (L \times B_k) = \emptyset \), whence \( \Pi_X[A_n \cap (X \times B_k)] \) is clopen and disjoint from \( L \). It follows that \( \Pi_X[A_n \cap (X \times B_k)] \) is countable because otherwise it would have some member of \( L \) as a limit point. Therefore,

\[
D = \bigcup_{n \in \omega} \bigcup_{k \in \omega} \Pi_X[A_n \cap (X \times B_k)]
\]

is a countable open subset of \( X \) that is disjoint from \( L \) and hence is clopen in \( X \).

However,

\[
D = \bigcup_{n \in \omega} \Pi_X[A_n \cap (X \times C)]
= \Pi_X[\bigcup_{n \in \omega} A_n \cap (X \times C)]
= \Pi_X[W \cap (X \times C)].
\]

Therefore,

(4) \( W \cap (X \times C) \subseteq (D \times \omega^*) \).

Now,

\[
W \cap (X \times \omega^*) = [W \cap (X \times C)] \cup [W \cap (X \times (\omega^* \setminus C))].
\]

Thus,

\[
C\mathcal{X}(W \cap (L \times C)) = (P \cup \{Q\}) \cap (L \times C)
= [P \cap (L \times C)] \cup [Q \cap (L \times C)],
\]

where

\[
P = C\mathcal{X}(W \cap (X \times C))
\]

and

\[
Q = C\mathcal{X}(W \cap (X \times (\omega^* \setminus C))).
\]

By (4), \( P \cap (L \times C) \subseteq [C\mathcal{X}(W \cap (L \times C))] \cap (L \times C) \subseteq [C\mathcal{X}(D \times \omega^*]) \cap (L \times C) \). The latter is equal to \( [C\mathcal{X}(D \times \omega^*]) \cap (X \times \omega^*) \cap (L \times C) - (D \times \omega^*) \cap (L \times C) \) because, as shown above, \( D \times \omega^* \) is closed in \( X \times \omega^* \). Also, \( Q \subseteq X \times (\omega^* \setminus C) \) because the latter is closed in \( X \times \omega^* \).

Now, by the above,

\[
P \cap (L \times C) \subseteq (D \times \omega^*) \cap (L \times C) = \emptyset
\]

and

\[
Q \cap (L \times C) \subseteq X \times (\omega^* \setminus C) \cap (L \times C) = \emptyset.
\]

Therefore, we may conclude that (2) holds.

To complete the proof that \( U \) is substonean, first note that \( L \times C = \bigcup_{n \in \omega} L \times B_n \) is a countable union of Lindelöf spaces and hence is a Lindelöf space. Thus, \( W \cup (L \times C) \) is a Lindelöf space and hence is \( C^* \)-embedded in the \( F \)-space \( K \) (see Dow²).
Now, \( L \times C \) is closed in \( W \cup (L \times C) \) because \( W \) is open in \( K \). By (2), \( C_{\alpha}W \cap (L \times C) = \emptyset \), so \( W \) is also closed in \( W \cup (L \times C) \). Consequently, \( W \) and \( L \times C \) are disjoint clopen subsets of the \( C^* \)-embedded subspace \( W \cup (L \times C) \) of \( K \) and hence \( C_{\alpha}W \cap C_{\alpha}(L \times C) = \emptyset \). Therefore, \( C_{\alpha}W \subseteq U \) and we know that \( C_{\alpha}K = C_{\alpha}W \) is compact. This completes the proof that \( U \) is a substonean space that fails to be an \( F \)-space.

**Remark.** A. Dow\(^2\) notes that \( |C^*(K)| = \omega \cdot \omega_2 \). Hence, if \( c = \omega_2 \), then the space \( U \) is an open subspace of \( K \) that violates the conclusion of Proposition 4. Therefore, the assumption that CH holds is essential for it to be valid.

Recall that a point of a space \( X \) is called a weak \( P \)-point if \( q \) is not in the closure of any countable subset of \( X \setminus \{q\} \). It follows immediately from Theorems 1 and 3 that:

- If \( q \) is a weak \( P \)-point of a compact \( F \)-space and if \( q \) fails to be a \( P \)-point, then \( q \) is in the closure of a \( \omega \)-compact subspace of \( X \setminus \{q\} \), but it is not in the closure of any of its countable subspaces.

As is noted in reference 19, \( \omega^* \) contains such points.

Now, recall from reference 8 (problem 11) that a space \( X \) is called *basically disconnected* if the closure of any cozero set of \( X \) is open. In Grove and Pedersen,\(^5\) a locally compact space is called a *Rickart space* if the closure of each of its \( \sigma \)-compact open sets is open and compact. Clearly, every Rickart space is substonean. The authors say on page 132 of the latter: "In [Gillman and Jerison, problem 11H], these spaces are said to be basically disconnected." An infinite discrete space is basically disconnected without being a Rickart space. On the other hand, because each point of a Rickart space has a compact open neighborhood that is a Rickart space, because every Rickart space is basically disconnected, and because locally basically disconnected spaces are basically disconnected [as noted in reference 12 (4.6)], it follows that every Rickart space is basically disconnected.

We are indebted to A. Dow and R. Levy for pointing this out to us. Of course, this argument fails in the case of substonean spaces and \( F \)-spaces because locally \( F \)-spaces need not be \( F \)-spaces; see, for example, Dow.\(^3\)

The proof of the following result is similar to that of Theorem 1 and is an exercise.

**Theorem 6.** The following properties of a locally compact \( X \) are equivalent:

(a) \( X \) is a Rickart space;
(b) \( \alpha X \) is basically disconnected and \( X \) is compact or \( \infty \) is a \( P \)-point of \( \alpha X \);
(c) there is a basically disconnected compactification space \( Y \) of \( X \) such that \( X = Y \setminus \{\alpha Y \} \) is a \( P \)-set of \( Y \).

In problem 2.2 of reference 9, Grove and Pedersen pose the problem of characterizing those substonean spaces that are closed subspaces of Rickart space. The authors say that if (CH) holds, then every totally disconnected substonean space of weight no larger than \( c \) is homeomorphic to a closed subspace of \( \beta \omega \) (see reference 14 or 19). Presumably, they intend to pose this problem only for compact spaces. It has been considered before and the only other known result is as follows: if Martin's axiom holds and \( c = \omega_2 \), then there is a compact totally disconnected \( F \)-space that is not a closed
Having described the relationship between substonean spaces and $F$-spaces and between Rickart spaces and basically disconnected spaces, we devote the rest of this paper to miscellaneous remarks about some of the results of Grove and Pedersen.\footnote{Grove and Pedersen call a subset $S$ of a topological space $X$ \textit{basically isolated} if $Y \cap S = \emptyset$ implies $C_X Y \cap C_X S = \emptyset$ for every $\sigma$-compact open space of $Y$ of $X$.}

**Proposition 7.** A nonempty closed subset $S$ of a locally compact space $X$ is basically isolated if and only if it is a $P$-set.

\textit{Proof:} Suppose $S$ is basically isolated, $C$ is a cozero set of $X$, and $C \cap S = \emptyset$. Suppose also that there is an $x \in C_X C \cap S$. Because $X$ is locally compact, $x$ has a $\sigma$-compact open neighborhood $U$, and because $C$ is an $F_\sigma$, the set $U \cap C$ is $\sigma$-compact. Thus, $C_X (U \cap C) \cap S = \emptyset$ because $S$ is basically isolated. If $W$ is any neighborhood of $x$, then $x \in (W \cap U) \cap C \cap W \cap (U \cap C)$ because $x \in C_X C$. However, if this is the case, $x \in C_X (U \cap C) \cap S$. This contradiction shows that $C_X C \cap S = \emptyset$, whence $S$ is a $P$-set.

The converse is immediate because a $\sigma$-compact open subset of $X$ is a cozero set.

In reference 9 (theorem 1.13), it is shown that if two substonean spaces are "glued" together by means of a continuous open surjection $\phi$ of a $P$-set of one of them onto a $P$-set of the other, then the resulting space is also substonean. In case these spaces are compact, lemma 1.4.1 of reference 19 shows that this conclusion holds under the weaker assumption that $\phi$ is continuous.

In proposition 1.15 of reference 9, it is shown that a substonean image of a product of compact metric spaces under a continuous open map is stonean. This result can easily be generalized as follows.

Recall that a space $X$ satisfies ccc (the \textit{countable chain condition}) if each family of a pairwise disjoint family of open subsets of $X$ is countable.

**Proposition 8.** An $F'$-space $\overline{X}$ that is a continuous image of a ccc-space $Y$ is extremally disconnected. In particular, if $X$ is either an $F$-space or a substonean space and is a continuous image of such a space $Y$, then $X$ is extremally disconnected.

\textit{Proof:} Clearly, a continuous image of a ccc-space satisfies ccc, and it is shown in Dow\footnote{Dow shows that any $F$-space satisfying ccc is an $F$-space. Therefore, the first part of the proposition follows from the fact that every $F$-space that satisfies ccc is extremally disconnected; see reference 15 (6L).} that any $F'$-space satisfying ccc is an $F$-space. Therefore, the first part of the proposition follows from the fact that every $F$-space that satisfies ccc is extremally disconnected; see reference 15 (6L). The second part follows from the fact that substonean spaces and $F$-spaces are $F'$-spaces.

Recall from reference 15 that a product of spaces satisfies ccc if each finite subproduct does. Hence, we have the following corollary.

**Corollary.** If $X$ is substonean and is a continuous image of the product of a family of separable spaces, then $X$ is extremally disconnected and is stonean if each factor in the product is compact.
In proposition 1.16 of reference 9, it is shown that every compact subspace of a locally compact space is finite. This is an immediate consequence of the fact that no infinite compact space is homogeneous; see reference 19 (3.4.2).

In section 3 of reference 9, the Stone-Čech remainder \( \beta X \setminus X = C^* \) of a locally compact space \( X \) is called the corona of \( X \), and a map \( \phi: X \to Y \) of a locally compact space into a locally compact space \( Y \) is called proper if it is continuous and inverse images of compact sets are compact. Such mappings are known to be closed; see Engelking (3.7.18).\(^5\) Mappings that are closed and continuous and such that point inverses are compact are usually called perfect maps. It is well known that if \( \phi: X \to Y \) is proper, then there is a continuous extension \( \beta \phi: \beta X \to \beta Y \) such that \( \phi[X]^* \subseteq Y^* \); see Henriksen and Isbell.\(^1\) The following is a restatement of proposition 3.1 in Grove and Pedersen,\(^9\) whose proof we were unable to follow.\(^9\)

**Proposition 10.** If \( \phi: X \to Y \) is a perfect map of a locally compact \( \sigma \)-compact space \( X \) into a locally compact \( \sigma \)-compact space \( Y \), then \( \beta \phi[X]^* \) is a \( \sigma \)-set of \( Y^* \).

It is true, though, and follows easily from the fact shown in reference 18 (lemma 2) that if \( A \) is a closed subspace of a locally compact \( \sigma \)-compact space \( X \), then \( \beta \phi[A] \cap X^* \) is a \( \sigma \)-set of \( X^* \).

Note that most of the remaining results in section 3 of Grove and Pedersen\(^9\) may be found in Walker.\(^21\)

In summary, we found (Grove and Pedersen\(^9\)) an interesting paper and we hope that the reader will find that our results enhance it and give insight into how general topology can be used as a tool in the study of functional analysis.

**REFERENCES**
