When is $C(X)/P$ a valuation ring for every prime ideal $P$?

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**Abstract**


A Tychonoff space $X$ is called an SV-space if for every prime ideal $P$ of the ring $C(X)$ of continuous real-valued functions on $X$, the ordered integral domain $C(X)/P$ is a valuation ring (i.e., of any two nonzero elements of $C(X)/P$, one divides the other). It is shown that $X$ is an SV-space iff $\nu X$ is an SV-space iff $\beta X$ is an SV-space. If every point of $X$ has a neighborhood that is an F-space, then $X$ is an SV-space. An example is supplied of an infinite compact SV-space such that any point with an F-space neighborhood is isolated. It is shown that the class of SV-spaces includes those Tychonoff spaces that are finite unions of $C^*$-embedded SV-spaces. Some open problems are posed.

**Keywords:** Tychonoff space, SV-space, F-space, $C^*$-embedded, Stone–Čech compactification, real-closed ring, valuation ring, minimal prime ideal, ring of continuous real-valued functions.

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1. **Introduction**

Throughout, $X$ will denote a Tychonoff space. In [1], Cherlin and Dickmann call a (commutative) integral domain $D$ real-closed if it is totally ordered, positive elements have square roots in $D$, each monic polynomial of odd degree in $D[x]$ has a zero in $D$, and if $a$, $b$ in $D$ satisfy $0 < a < b$, then $a$ divides $b$. They show that for any prime ideal $P$ of the ring $C(X)$ of continuous real-valued functions on $X$, $C(X)/P$ is real-closed if and only if it is a valuation ring; that is of any two elements in $C(X)/P$, one of them divides the other. A prime ideal $P$ is called real-closed if...
$C(X)/P$ is real-closed. By the above, if $P$ is an ideal of $C(X)$, $P$ is real-closed if and only if whenever $f, g \in C(X)$ and $0 < g < f \mod P$, there is a $k \in C(X)$ such that $(g - kf) \in P$. After noting that every maximal ideal of $C(X)$ is real-closed, Cherlin and Dickmann examine conditions on $X$ under which there are nonmaximal real-closed ideals.

While the focus in [1] is local, ours is global. We call $C(X)$ a survaluation ring, or an $SV$-ring if each of its prime ideals is real-closed. If $C(X)$ is an $SV$-ring, we call $X$ an $SV$-space. In the sequel, we study the nature of $SV$-spaces and we improve on some of the results in [1]. In particular we show that $X$ is an $SV$-space iff $\omega X$ is an $SV$-space iff $\beta X$ is an $SV$-space, that $C^*$-embedded subspaces of $SV$-spaces are $SV$-spaces, and that finite unions of compact $SV$-subspaces are $SV$-spaces. Spaces that are local $F$-spaces in the sense of [2] are $SV$-spaces but the converse fails in a very strong sense.

The reader should not regard this paper as a continuation of [1] and we do not pretend that our results improve greatly on those in it. Instead we proceed in a rather different direction and are seeking ultimately a topological characterization of the algebraically defined notion of an $SV$-space. We regard this paper as a first step.

We close with some remarks and problems.

2. $SV$-spaces

As in [2], $\omega X$ denotes the Hewitt real-compactification of $X$, while $\beta X$ denotes the Stone–Cech compactification of $X$; see [2] for unfamiliar definitions. Let $C^*(X)$ denote the subring of bounded elements of $C(X)$, and note that $C(\beta X)$ and $C^*(X)$ are isomorphic. The first proposition tells us that there is no loss of generality in concentrating on compact $SV$-spaces. To prove it, we will need the following facts established in [1].

(1) If $P$ is a real-closed ideal of $C(X)$ and $Q$ is a (proper) prime ideal containing $P$, then $Q$ is real-closed.

As noted in [1], it follows from the fact that $C(X)/Q$ is a homomorphic image of $C(X)/P$. (If $\phi(f + P) = f + Q$, then $\phi$ is a homomorphism with kernel $Q/P$.)

An immediate consequence of (1) is:

**Proposition 2.1.** $X$ is an $SV$-space if and only if every minimal prime ideal of $C(X)$ is real-closed.

**Proposition 2.2.** For any Tychonoff space $X$, the following are equivalent.

(a) $X$ is an $SV$-space.

(b) $\omega X$ is an $SV$-space.

(c) $\beta X$ is an $SV$-space.

**Proof.** The equivalence of (a) and (b) is immediate since $C(X)$ and $C(\omega X)$ are isomorphic.
In the proof of [3, Theorem 5.1], it is shown that the map \( P \to P \cap C^*(X) \) is a surjection of the set of minimal prime ideals of \( C(X) \) onto the set of minimal prime ideals of \( C^*(X) \), and in the proof of [1, Corollary 12], it is shown that \( P \) is a real-closed ideal of \( C(X) \) if and only if \( P \cap C^*(X) \) is a real-closed ideal of \( C^*(X) \). Since \( C(\beta X) \) and \( C^*(X) \) are isomorphic, this establishes the equivalence of (a) and (c). So the proposition holds. □

The next result is a minor modification of [1, Theorem 1], but its proof seems somewhat different.

Before stating it, we recall some general information from [2]. If \( f \in C(X) \), then let \( Z(f) = \{ x \in X : f(x) = 0 \} \) denote the zerorset of \( f \), and \( \text{coz } f = X \setminus Z(f) \) denote its cozerorset. Recall that a subspace \( Y \) of \( X \) is said to be \( C \)-embedded (respectively \( C^* \)-embedded) in \( X \) if the map that sends each \( f \) in \( C(X) \) (respectively in \( C^*(X) \)) to its restriction to \( Y \) is a surjection. Also, an ideal \( I \) of \( C(X) \) is called a \( z \)-ideal if whenever \( Z(f) = Z(g) \) and \( g \in I \), then \( f \in I \). Not every prime ideal is a \( z \)-ideal as noted in [2, Chapter 2], but every minimal prime ideal of \( C(X) \) is a \( z \)-ideal; see [3].

**Theorem 2.3.** If \( P \) is a prime \( z \)-ideal of \( C(X) \), then \( P \) is real-closed if and only if:

For each \( f \in C(X) \) and \( F \in C^*(\text{coz } f) \), there is a \( w \in P \) such that \( F|_{\text{coz } f \cap Z(w)} \) has a continuous extension over \( X \).

\( (*) \)

**Proof.** Suppose first that \( P \) is real-closed. It suffices to show that \( (*) \) holds for those \( f \in C(X) \) such that \( f \uparrow 0 \). Pick \( F \) in \( C^*(\text{coz } f) \) and assume without loss of generality that \( 0 \leq F \leq 1 \).

Let \( h(x) = F(x)f(x) \) if \( x \in \text{coz } f \), and let \( h(x) = 0 \) if \( x \in Z(f) \). Then \( h \in C^*(X) \) and \( 0 \leq h(x) \leq f(x) \) for all \( x \in X \). Since \( C(X)/P \) is a valuation ring, there is a \( w \in P \) and a \( k \in C(X) \) such that \( h = kf \) on \( Z(w) \). Since \( h = Ff \) on \( \text{coz } f \), it follows that \( k \) is a continuous extension of the restriction of \( F \) to \( \text{coz } f \cap Z(w) \) over \( X \). Hence \( (*) \) holds.

Suppose conversely that \( (*) \) holds for a prime ideal \( P \) of \( C(X) \), and \( 0 \leq g \leq f \) mod \( P \). Then \( \{ x \in X : 0 \leq g(x) \leq f(x) \} = Z(u) \) for some \( u \in P \). Thus \( f \uparrow g = f \) on \( Z(u) \), so, since \( P \) is a \( z \)-ideal, we may assume without loss of generality that \( 0 \leq g \leq f \) on \( X \). For this \( f \), let \( k = g/f \) on \( \text{coz } f \). By \( (*) \), there is a \( w \in P \) such that the restriction of \( k \) to \( Z(w) \) has a continuous extension \( k \) over \( X \). Clearly \( g = kf \) mod \( P \), so \( P \) is real-closed. This completes the proof of the theorem. □

The next simple observation follows immediately from the definition of an SV-ring and Proposition 2.2.

**Proposition 2.4.** If \( C^*(X) \) is a homomorphic image of \( C^*(X) \) and \( X \) is an SV-space, then so is \( Y \). In particular, every \( C^* \)-embedded subspace of an SV-space is an SV-space.

It is shown in [1] that without making special set-theoretic assumptions, it is impossible to establish the existence of a nonmaximal real-closed ideal of \( C(X) \) if
X is a nondiscrete metrizable space, and that the one-point compactification $aN = N \cup \{\infty\}$ of the countable discrete space $N$ cannot be an SV-space since not every minimal prime ideal is real-closed. So it follows from Proposition 2.4 that we have:

**Corollary 2.5.** An infinite SV-space contains no nontrivial convergent sequence.

Recall from [2, Chapter 14] that $X$ is called an F-space if every cozeroset of $X$ is $C^*$-embedded or, equivalently if finitely generated ideals of $C(X)$ are principal. It is noted there that if $X$ is an F-space, then $C(X)/P$ is a valuation ring for every prime ideal $P$. Thus every F-space is an SV-space. More generally, it follows immediately from Theorem 2.3 that:

**Corollary 2.6.** If $P$ is a prime z-ideal of $C(X)$ that contains a w such that $Z(w)$ is a $C^*$-embedded F-space, then $P$ is real-closed.

The next result appears also in [1] and follows immediately from the fact that every prime ideal of $C(X)$ that is contained in the maximal ideal $M_p = \{f \in C(X): f(p) = 0\}$ contains, for each $p \in X$, the ideal $O_p = \{f \in C(X): p \in \text{Int} Z(f)\}$, and that every maximal ideal of $C(X)$ takes this form if $X$ is compact; see [2, Chapter 4].

**Corollary 2.7.** If $X$ is compact and $p \in X$ has a neighborhood that is an F-space, then any prime ideal contained in $M_p$ is real-closed. Hence, if every point of $X$ has a neighborhood that is an F-space, then $X$ is an SV-space.

Recall that a point $p$ of a space $X$ is called a P-point if for each $f \in C(X)$, $f(p) = 0$ implies $p \in \text{Int} Z(f)$, and $X$ is called a P-space if each of its points is a P-point. As is well known, a compact P-space is finite; see [2, Chapter 14]. In [1], an example is given of a compact SV-space with a point that has no neighborhood that is an F-space. Below, we derive that example and others to show that an SV-space can depart "sharply" from being an F-space. First we prove:

**Theorem 2.8.** If a Tychonoff space $X$ is a finite union of C-embedded SV-spaces, then $X$ is an SV-space. In particular, a compact space that is the union of finitely many closed SV-subspaces is an SV-space.

**Proof.** It suffices to prove the theorem in case $X = X_1 \cup X_2$ is the union of two C-embedded SV-spaces. If $f \in C(X)$, let $\phi_i(f)$ denote the restriction of $f$ to $X_i$ and observe that $\ker \phi_i = \{g \in C(X): g(X_i) = \{0\}\}$ for $i = 1, 2$. If $P$ is a prime ideal of $C(X)$, then since $(\ker \phi_1)(\ker \phi_2) \subseteq (\ker \phi_1) \cap (\ker \phi_2) = \{0\}$, $P$ contains one of these two ideals, say $\ker \phi_1$, and hence $P_1 = P/\ker \phi_1$ is a prime ideal of the ring $C(X)/\ker \phi_1$. Since $X_1$ is C-embedded in $X$, this latter ring is isomorphic to $C(X_1)$. Because the rings $C(X)/P$ and $(C(X)/\ker \phi_1)/(P/\ker \phi_1)$ are isomorphic, it follows that $C(X)/P$ and $C(X_1)/P_1$ are isomorphic; since $X_1$ is an SV-space $C(X_1)/P_1$ is
a valuation domain. Since \( P \) is an arbitrary prime ideal of \( C(X) \), \( X \) must be an SV-space. □

This enables us to give:

**Example 2.9.** A compact SV-space that fails to have an \( F \)-space neighborhood at any nonisolated point.

Let \( T \) denote the topological sum of two copies of \( \beta N \), and let \( X \) be obtained by identifying corresponding points of \( \beta N \setminus N \). Since \( \beta N \) is an \( F \)-space, it follows from the last theorem that \( X \) is an SV-space. Clearly each nonisolated point is in the closure of two (discrete) cozerosets, and hence has no neighborhood that is an \( F \)-space.

The example given in [1] of an SV-space that fails to be a local \( F \)-space is obtained by identifying two copies of a compact \( F \)-space at a non-\( P \)-point.

Much remains to be learned about SV-spaces, as the next and final section shows.

3. Remarks and problems

**Remark 3.1.** While the focus of the study of prime ideals of this paper is global, there are some local problems that seem pertinent. In [5] and in [1, Section 2.4], a point \( p \) of \( \beta X \) is called a \( \beta F \)-point if \( C(X)/O^p \) is a prime ideal (where \( O^p = \{ f \in C(X) : \text{there is a neighborhood } V \text{ of } p \text{ in } \beta X \text{ such that } Z(f) \supseteq V \cap X \} \)). It is shown in [2, Chapter 14] that \( X \) is an \( F \)-space if and only if every point of \( X \) is a \( \beta F \)-point. It is shown in Lemma 2 and Corollary 6 of [1, Section 2.4] that if \( p \) is a \( \beta F \)-point of \( X \), and \( f \in C(X) \), \( \text{coz}(f) \) is \( C^* \)-embedded in \( (\text{coz}(f)) \cup \{ p \} \). So, if \( \{ p \} \) is also a \( G_s \), then \( X \setminus \{ p \} \) is \( C^* \)-embedded in \( X \), and it follows that if \( X \) is also compact, then \( p \) is isolated.

While some sufficient conditions are given in [1] for \( O^p \) to be real-closed, the following questions remain open.

**Question 3.2.** Suppose \( X \) is any (compact) space and \( O_p \) is prime. Does it follow that it is real-closed?

**Question 3.3.** Is there an SV-space with a \( G_s \)-point \( p \) such that \( O_p \) is prime but no neighborhood of \( p \) is an \( F \)-space?

**Question 3.4.** Is there a "global" version of Theorem 2.3? That is, is there a characterization of SV-spaces that does not refer to individual prime ideals?

**Comment 3.5.** At one time we had conjectured that for any prime \( z \)-ideal \( P \) of \( C(X) \) (where \( X \) is a Tychonoff space) \( P \) is real-closed if and only if:

For each \( f \in C(X) \), there is a \( w \in P \) such that \( \text{coz}(f) \cap Z[w] \) is \( C^* \)-embedded in \( X \).

\((\ast)\)
Clearly Theorem 2.3 implies that if (\*) holds, then \( P \) is real-closed.

The referee observed that the converse fails, at least if Martin's axiom holds. For, in this case \( Y = \beta N \setminus N \) has a \( P \)-point \( p \). Let \( U = \{ K \subset N : p \in \text{Cl}_{\beta N} K \} \), and let

\[
P_U = \{ f \in C(\alpha N) : Z(f) \cap N \in U \}.
\]

In [1, Theorem 1, Section 3.1] it is shown that \( P_U \) is a (minimal prime) real-closed ideal of \( C(\alpha N) \). If \( j \) denotes the reciprocal of the identity function on \( N \), then it is clear that \( \text{coz} j \cap Z[w] = Z[w] \) is not \( C^* \)-embedded in \( \alpha N \) for any \( w \in P_U \). Thus (\*) does not hold.

This counterexample is less than satisfying because of its dependence on Martin's axiom and because \( \alpha N \) is not an SV-space.

We call a space \textit{almost discrete} if it has exactly one nonisolated point. It is easy to verify that any almost discrete space is normal, and is perfectly normal if its nonisolated point is a \( G_\delta \).

\textbf{Question 3.6.} Which almost discrete spaces are SV-spaces?

\textbf{Note.} We have been able to show that an almost discrete space is an SV-space if and only if it is the union of finitely many closed basically disconnected subspaces; see [4].

\textbf{References}