The maximal regular ideal of some commutative rings

EMAD ABU OSBA, MELVIN HENRIKSEN, OSAMA ALKAM, F.A. SMITH

Abstract. In 1950 in volume 1 of Proc. Amer. Math. Soc., B. Brown and N. McCoy showed that every (not necessarily commutative) ring $R$ has an ideal $\mathfrak{M}(R)$ consisting of elements $a$ for which there is an $x$ such that $ax = a$, and maximal with respect to this property. Considering only the case when $R$ is commutative and has an identity element, it is often not easy to determine when $\mathfrak{M}(R)$ is not just the zero ideal. We determine when this happens in a number of cases: Namely when at least one of $a$ or $1 - a$ has a von Neumann inverse, when $R$ is a product of local rings (e.g., when $R$ is $\mathbb{Z}_n$ or $\mathbb{Z}_n[i]$), when $R$ is a polynomial or a power series ring, and when $R$ is the ring of all real-valued continuous functions on a topological space.

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1. Introduction

Throughout $R$ will denote a commutative ring with identity element 1 unless the contrary is stated explicitly, and the notation of [AHA04] will be followed.

1.1 Definition. An element $a \in R$ is called regular if there is a $b \in R$ such that $a = ab$. Let $\text{vr}(R) = \{ a \in R : a \text{ is regular} \}$ and $\text{nvr}(R) = R \setminus \text{vr}(R)$. An ideal $I$ of $R$ is called a regular ideal if $I \subseteq \text{vr}(R)$. The element $a$ is called $m$-regular if the ideal generated by $a$ is a regular ideal. Let $\mathfrak{M}(R) = \{ a \in R : a \text{ is } m\text{-regular} \}$. A ring $R$ is called von Neumann regular ring (VNR ring) if $R = \text{vr}(R)$.

This terminology is motivated in part by a theorem of Brown and McCoy in which they show that $\mathfrak{M}(R)$ is a regular ideal. Indeed it is the largest regular ideal or $R$. See [BM50]. $R$ may contain regular elements which are not $m$-regular, as one can see easily that $3 \in \text{vr}(\mathbb{Z}_4) \setminus \mathfrak{M}(\mathbb{Z}_4)$. (As usual, $\mathbb{Z}_n$ denotes the ring $\mathbb{Z}$ of integers mod $n$ for a positive integer $n$.)

If $S \subset R$, then $\text{Ann}(S)$ denotes $\{ a \in R : aS = \{0\} \}$, the set of maximal ideals of $R$ is denoted by $\text{Max}(R)$, and their intersection $J(R)$ is the Jacobson radical of $R$. In [BM50], the following is also established.
1.2 Lemma.

\(\mathfrak{M}(R/\mathfrak{M}(R)) = \{0\}\).
\(\mathfrak{M}(R) \cap J(R) = \{0\}\).
\(\mathfrak{M}(R) \subset \text{Ann}(J(R))\).
\(\mathfrak{M}(R) \cap \text{Ann}(\mathfrak{M}(R)) = \{0\}\).

If \(R/J(R)\) is VNR-ring, then \(\mathfrak{M}(R) = \{0\}\) if and only if \(\text{Ann}(J(R)) \subset J(R)\).
If \(R\) satisfies the descending chain condition on ideals, then \(R = \mathfrak{M}(R) + \text{Ann}(\mathfrak{M}(R))\).

For each ideal \(I\) of \(R\), let \(mI = \{a \in I : a \in aI\}\) = \(\{a \in R : I + \text{Ann}(a) = R\}\).
Then \(mI\) is called the pure part of \(I\). An ideal \(I\) is called a pure ideal if \(I = mI\). It is clear that \(a \in mM\) for an \(M \in \text{Max}(R)\), if and only if \(\text{Ann}(a)\) is not contained in \(M\).

The following description of \(\mathfrak{M}(R)\) will be used frequently below.

1.3 Theorem. If \(R\) is not a von Neumann regular ring, then \(\mathfrak{M}(R) = \bigcap \{mM : M \in \text{Max}(R)\} \neq m\mathfrak{M}(R)\) is the intersection of the pure parts of those maximal ideals \(M\) of \(R\) that are not pure.

PROOF: If \(a \notin \mathfrak{M}(R)\), then there is an \(x \in R\) such that \(ax \notin \text{vr}(R)\). So by Theorem 2.4 of [AHA04], there is an \(N \in \text{Max}(R)\) such that \(ax \in N \setminus mM\). It follows that \(N\) is not pure and \(a \notin \bigcap \{mM : M \in \text{Max}(R)\}\). Thus \(\bigcap \{mM : M \in \text{Max}(R)\} \neq m\mathfrak{M}(R)\).

If instead \(a \in \mathfrak{M}(R)\) and there is an \(M \in \text{Max}(R)\) and an \(x \in M \setminus mM\), then \(ax \in mM\) and so as noted above, there is a \(b \notin M\) such that \(bax = 0\). So \(ba \in \text{Ann}(x)\) which is contained in \(M\) because this maximal ideal in not pure. But \(M\) is a prime ideal, so \(a \notin M\). Thus \(\mathfrak{M}(R) \subset mM\). Hence \(\mathfrak{M}(R) \subset \bigcap \{mM : M \in \text{Max}(R)\}\). \(\square\)

In this article, we determine when \(\mathfrak{M}(R)\) is not the zero ideal for a number of classes of rings. In Section 2, we study rings in which at least one of \(a\) or \(1 - a\) has a von Neumann inverse. Section 3 is devoted to the study of products of local rings (e.g., the ring \(\mathbb{Z}_n\) of integers modulo an integer \(n \geq 2\) and to \(\mathbb{Z}_n[i]\)). The complicated conditions needed to describe when \(\mathfrak{M}(\mathbb{Z}_n[i]) \neq \{0\}\) hint at why it may be quite difficult to describe when the maximal regular ideal of a finite ring is nonzero. In Section 4, it is shown that the maximal regular ideal of a polynomial or powers series ring is the zero ideal, and in Section 5, it is determined when the maximal regular ideal of the ring of all continuous functions on a topological space is nonzero.

2. Von Neumann local and strong von Neumann local rings

Recall from [AHA04] that \(R\) is called a von Neumann local (VNL) ring if \(a \in \text{vr}(R)\) or \(1 - a \in \text{vr}(R)\) for each \(a \in R\). It is easy to see that VNR rings and local rings are VNL rings. \(R\) is called a strong von Neumann local (SVNL) ring if
whenever the ideal \( \langle S \rangle \) generated by a subset \( S \) of \( R \) is all of \( R \), then some element of \( S \) is in \( \text{vr}(R) \), or equivalently if \( \langle \text{vvr}(R) \rangle \neq R \). Clearly every SVNL ring is a VNL ring, but the validity of the converse remains an open problem. \( R \) is called a Gelfand ring or a PM ring if each of its proper prime ideals is contained in a unique maximal ideal. If \( M \) is a maximal ideal of \( R \), then \( O_M \) denotes intersection of all of the (minimal) prime ideals of \( R \) that are contained in \( M \).

2.1 Lemma. Every VNL ring \( R \) is a Gelfand ring and if \( R \) is also reduced, then \( mM = O_M \) whenever \( M \in \text{Max}(R) \).

Proof: The first assertion is shown in [C84]. (Combine in that paper Proposition 4.4, Theorems 3.2 and 2.4 with Proposition 1.1.) The second assertion is shown in Proposition 3 of [H77].

See also [D071].

Next, we make use of Theorem 1.1 above.

In Theorem 2.6 of [AHA04] it is shown that \( R \) is an SVNL ring that is not a VNR ring if and only if it has exactly one maximal ideal that fails to be pure. Combining this with Theorem 1.3 yields:

2.2 Theorem. If \( R \) is an SVNL ring that is not a VNR ring, then it has a unique maximal \( N \) that is not pure. Moreover \( \mathfrak{m}(R) = mN = O_M \).

Proof: The first assertion is part of Theorem 2.6 of [AHA04], and the second is immediate from Theorem 1.3 and Lemma 2.1.

Next we begin to exhibit a class of rings whose maximal regular ideal is not the zero ideal.

2.3 Lemma. If \( R \) and \( S \) are commutative rings with identity whose direct sum \( R \oplus S \) is a VNL ring, then at least one of \( R \) and \( S \) is a VNR ring.

Proof: Suppose instead that there are \( r \in R \) and \( s \in S \) that are not von Neumann regular. Then neither \( (r, 1-s) \) nor \( (1, 1) - (r, 1-s) = (1-r, s) \) are von Neumann regular in \( R \oplus S \), so the conclusion follows.

2.4 Theorem. If \( R \) is a VNL ring that is neither local nor a VNR ring, then \( \mathfrak{m}(R) \) contains \( fR \) for some idempotent \( f \) not in \( \{0, 1\} \) and hence is not the zero ideal.

Proof: By Theorem 4.6 of [AHA04], a nonlocal VNL ring has an idempotent \( e \notin \{0, 1\} \), so \( R = eR \oplus (1-e)R \). Thus by Lemma 2.3, exactly one of these two summands must be a VNR ring, which is a nonzero ideal included in \( \mathfrak{m}(R) \).

3. Products of local rings

In this section, it will be determined when a direct product of local rings has a nonzero maximal regular ideal.
It is an exercise to show that a local VNR ring is a field. Moreover, if $M$ is the unique maximal ideal of $R$, and $a = am \in mM$ for some $m \in M$, then $a = 0$ since $1 - m$ is invertible. Because each element of $\mathfrak{M}(R)$ is in $mM$, we conclude from Theorem 1.3 that:

3.1 Lemma. If $R$ is a local ring, then $R$ is a field or $\mathfrak{M}(R) = \{0\}$.

3.2 Lemma. If $R = \prod_{i \in I} R_i$ is the direct product of rings $R_i$ with identity, then:

1. $(r_i)_{i \in I} \in \mathfrak{v}(R)$ if and only if $r_i \in \mathfrak{v}(R_i)$ for each $i \in I$, and
2. $(r_i)_{i \in I} \in \mathfrak{M}(R)$ if and only if $r_i \in \mathfrak{M}(R_i)$ for each $i \in I$.

Proof: (1) $(r_i)_{i \in I} \in \mathfrak{v}(R)$ if and only if there exists $(x_i)_{i \in I} \in R$ such that $(r_i)_{i \in I} = (r_i x_i)_{i \in I} = (r_i x_i)_{i \in I}$ if and only if $r_i = r_i x_i$ for each $i \in I$ if and only if $r_i \in \mathfrak{v}(R_i)$ for each $i \in I$.

(2) Suppose that $(r_i)_{i \in I} \in \mathfrak{M}(R)$. Pick $r_k \in R_k$ and let $z \in R_k$.

Define $x_i = \begin{cases} z & i = k \\ 0 & i \neq k \end{cases}$.

Now, $(r_i)_{i \in I}(x_i)_{i \in I} \in \mathfrak{v}(R)$, so there exists $(y_i)_{i \in I} \in R$ such that $(r_i)_{i \in I}(x_i)_{i \in I} = ((r_i)_{i \in I}(x_i)_{i \in I}) y_i = ((r_i x_i)_{i \in I} y_i)_{i \in I}$. In particular, $r_k x = (r_k x)^2 y_k$. Thus $r_k \in \mathfrak{M}(R_k)$. Conversely, suppose that $r_i \in \mathfrak{M}(R_i)$ for each $i \in I$. Let $(x_i)_{i \in I} \in R$.

Then $r_i x_i \in \mathfrak{v}(R_i)$ for each $i \in I$, which implies that there exists $y_i \in R_i$ such that $r_i x_i = (r_i x_i)^2 y_i$ for each $i \in I$. Hence $(r_i)_{i \in I}(x_i)_{i \in I} = ((r_i x_i)_{i \in I}) y_i = ((r_i)_{i \in I}(x_i)_{i \in I}) y_i$ which implies that $(r_i)_{i \in I} \in \mathfrak{M}(R)$. \hfill $\square$

It follows that:

3.3 Theorem. If $R = \prod_{i \in I} R_i$ is the direct product of rings $R_i$ with identity, then $\mathfrak{M}(R) = \prod_{i \in I} \mathfrak{M}(R_i)$.

Because a local VNR ring is a field and if $R$ is a field, then $R = \mathfrak{M}(R)$, it follows that:

3.4 Corollary. If $R = \prod_{i \in I} R_i$ is the direct product of local rings $R_i$ with identity, then $\mathfrak{M}(R) \neq \{0\}$ if and only if $R_j$ is a field for at least one $j \in I$.

In Chapter VI of [M74], it is shown that every finite commutative ring with identity element is a direct product of local rings. Hence we have

3.5 Theorem. If $R$ is finite, then $\mathfrak{M}(R) \neq \{0\}$ if and only if $R$ is a direct product of local rings at least one of which is a field.

Much more is said about finite local rings in [M74]. If $R$ is such a ring then its unique maximal ideal $M$ is nilpotent and $\mathfrak{M}(R) = \{0\}$ by Lemma 3.1. Indeed, every element of $R$ is either nilpotent or invertible.

Next, some examples are considered.

It is well known that if $n > 1$ is in $\mathbb{Z}$, then $\mathbb{Z}_n$ is local if and only if $n = p^k$ for some prime $p$ and positive integer $k$, and is a field if and only if $k = 1$. 
3.6 Corollary. If \( n = \prod_{i=1}^{s} p_i^{k_i} \) is the prime power decomposition of the positive integer \( n \), then \( Z_n \) is the direct product of the local rings \( \mathbb{Z}_{p_i^{k_i}} \), and \( \mathfrak{M}(R) \neq \{0\} \) if and only if \( k_j = 1 \) for at least one \( j \in \{1, \ldots, s\} \).

3.7 Definition. If \( i^2 = -1 \) and \( Z[i] = \{a + ib : a, b \in \mathbb{Z}\} \) is the ring of Gaussian integers, then for any integer \( n > 1 \), \( Z_n[i] = Z[i]/nZ[i] = \{a + ib : a, b \in Z_n\} \) denotes the ring of Gaussian integers \( \mod n \).

3.8 Lemma. (a) If an element \( a + ib \) of \( Z_n[i] \) is nilpotent [resp. idempotent]
then \( a^2 + b^2 \) is nilpotent [resp. idempotent] in \( Z_n \).
(b) \( a + ib \) is a unit in \( Z_n[i] \) if and only if \( a^2 + b^2 \) is a unit of \( Z_n \).
(c) \( (a + ib)^2 = a + ib \) is a nontrivial idempotent if and only if \( a^2 - b^2 = a \) and \( 2ab = b \) in \( Z_n \).

Proof: (a) If \( a + ib \) is nilpotent, then so is \( (a - ib)(a + ib) = a^2 + b^2 \) because complex conjugation is an automorphism of \( Z_n[i] \). The proof for idempotents is similar.
(b) follows because \( (a - ib)(a + ib) = a^2 - b^2 \) and any divisor of a unit is a unit.
(c) is an exercise. \( \square \)

As in Corollary 3.6, if \( n = \prod_{i=1}^{s} p_i^{k_i} \) is the prime power decomposition of the positive integer \( n \), then \( Z_n[i] \) is the direct product of the rings \( \mathbb{Z}_{p_i^{k_i}}[i] \). So by Theorem 3.3, \( \mathfrak{M}(Z_n[i]) = \prod_{i=1}^{s} \mathfrak{M}(Z_{p_i^{k_i}}[i]) \neq \{0\} \) if and only if at least one of the ideals in this latter product is nonzero. This motivates the question:

(*) If \( p \) and \( k \) are positive integers and \( p \) is prime, when is \( \mathfrak{M}(Z_{p^k}[i]) \neq \{0\} \)?

While it is true that \( Z_n \) is a local ring whenever \( n \) is a power of a prime, this is not the case for \( Z_n[i] \) as will be shown next. Recall that if a ring \( R \) is finite, then \( R \) is local if and only if its only idempotents are 0 and 1 (which are called trivial idempotents).

3.9 Theorem. If \( m = p^k \) for some prime \( p \) and positive integer \( k \), then \( Z_m[i] \) is local if and only if \( p = 2 \) or \( p \equiv -1(\mod 4) \).

Proof: We will show that if \( a + ib \) is a nontrivial idempotent of \( Z_m[i] \), then

(i) \( 2a \equiv 1(\mod p^k) \), and

(ii) there is a \( c \) such that \( c^2 \equiv -1(\mod p^k) \).

To see (i), recall from Lemma 3.8 that if \( a + ib \) is an nontrivial idempotent, then \( a^2 - b^2 = a \) and \( 2ab = b \) in \( Z_m \) and neither \( a \) nor \( b \) is \( 0(\mod p^k) \). This latter equation says \( b(2a - 1) \equiv 0(\mod p^k) \). By Lemma 3.8, \( c^2 + b^2 \) is an idempotent in \( Z_m \) and hence is congruent to 0, so if \( p \mid b \), then \( p \mid a \). It follows that \( p^k \mid b \) because \( 2ab = b \). A routine induction yields \( p^k \mid b \) and hence that \( b \equiv 0(\mod p^k) \); contrary to the assumption that \( a + ib \) is a nontrivial idempotent. Hence \( p \) is not a divisor of \( b \), i.e. \( b \) is a unit in \( Z_m \), but \( b(2a - 1) \equiv 0(\mod p^k) \). So (i) holds.
This shows that there are no nontrivial idempotents in $\mathbb{Z}_{2^k}[i]$. So this ring is also a field because it contains the nonzero nilpotent ideal $(1 + i)\mathbb{Z}_{2^k}[i]$. Thus $\mathfrak{M}(\mathbb{Z}_{2^k}) = \{0\}$ for all $k$.

Assume next that $p$ is odd and note that by (i) and its proof $(2b)^2 = 4(a^2 - a) \equiv (2a)^2 - 2(2a) = (p^k + 1)^2 - 2(p^k + 1) \equiv -1(\text{mod } p^k)$. So $c = 2b$ is the solution of the equation in (ii). Thus $\mathbb{Z}_m[i]$ has a nontrivial idempotent exactly when the equation in (ii) has a solution in which case $\frac{1}{2} + i\frac{c}{2}$ is such an idempotent.

It is noted in Chapter 5 of [L58] that for $p$ odd, the congruence $c^2 \equiv -1(\text{mod } p^k)$ has a solution, i.e. $-1$ is a quadratic residue mod $p^k$, when $p$ is odd if and only if it has one for $k = 1$. It is shown that $-1$ is a quadratic residue mod $p$ if and only if $p \equiv 1(\text{mod } 4)$. This completes the proof of the theorem. □

For a more thorough discussion of the topic of the last paragraph, see Section 5.3 of [L58].

3.10 Corollary. If $p$ is an odd prime, then $\mathbb{Z}_p[i]$ is a VNR ring.

PROOF: If $p \equiv -1(\text{mod } 4)$, then $\mathbb{Z}_p[i]$ is a field because by Theorem 7.2 of [L58], the congruence $a^2 + b^2 \equiv 0(\text{mod } p)$ has no solution. Assume next that $p \equiv 1(\text{mod } 4)$. It follows by Theorem 3.9 that $\mathbb{Z}_p[i]$ is not local, thus $\mathbb{Z}_p[i]$ (which has $p^3$ elements) is product of exactly two local rings, each isomorphic to $\mathbb{Z}_p$. Hence $\mathbb{Z}_p[i]$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ a product of two VNR rings. □

3.11 Corollary. If $m = p^k$ for some odd prime $p$ and positive integer $k$, then $\mathfrak{M}(\mathbb{Z}_m[i]) \neq \{0\}$ if and only if $k = 1$.

PROOF: As noted in the proof of Theorem 3.9, $\mathfrak{M}(\mathbb{Z}_{2^k}[i]) = \{0\}$ for all $k$. By the last corollary, if $p$ is an odd prime and $k = 1$, then $\mathfrak{M}(\mathbb{Z}_m[i]) \neq \{0\}$.

Now if $k > 1$ and $p \equiv -1(\text{mod } 4)$ or if $p = 2$, then by Theorem 3.9, $\mathbb{Z}_m[i]$ is a local ring which is not a field. So $\mathfrak{M}(\mathbb{Z}_m[i]) = \{0\}$ by Lemma 3.1.

If $k > 1$, $p \equiv 1(\text{mod } 4)$, and $a+ib$ is a nonunit of $\mathbb{Z}_m[i]$, then $a^2 + b^2 \equiv 0(\text{mod } p)$. If $p \mid a$, or $p \mid b$, then $p$ divides the other, so $p \mid (a + ib)$. Thus $a + ib$ is a nonzero nilpotent element of $\mathbb{Z}_m[i]$ since $k > 1$. If, instead $p$ fails to divide $a$ or $b$, then it is easy to verify that $p(a + ib)$ is a nonzero nilpotent in $\mathbb{Z}_m[i]$. Thus no nonzero nonunit of $R$ can be $m$-regular, and the existence of the nonzero nilpotent ideal $pR$ shows that no unit of $\mathbb{Z}_m[i]$ can be $m$-regular. Hence $\mathfrak{M}(\mathbb{Z}_m[i]) = \{0\}$ and the proof is complete. □

In summary we have using Theorem 3.3 and the above:

3.12 Corollary. If $n = \prod_{i=1}^s p_i^{k_i}$ is the prime power decomposition of the positive integer $n$, then $\mathfrak{M}(\mathbb{Z}_n[i]) \neq \{0\}$ if and only if $p_j$ is an odd prime and $k_j = 1$ for at least one $j \in \{1, \ldots, s\}$. 
4. Polynomial and power series rings

For each ring $R$, we write the polynomial ring as $R[x] = \{ \sum_{i=0}^{n} a_{i} x^{i} : a_{i} \in R \}$ and the power series ring by $R[[x]] = \{ \sum_{i=0}^{\infty} a_{i} x^{i} : a_{i} \in R \}$ where addition is coefficientwise, and in each case $(\sum a_{i} x^{i})(\sum b_{j} x^{j}) = \sum c_{k} x^{k}$, where $c_{k} = \sum_{i+j=k} a_{i} b_{j}$. The coefficient of $x^{k}$ in $c(x) = \sum c_{k} x^{k}$ is denoted by $c_{k}$. Both of these rings are commutative and have an identity. The next lemma is well known. See the first set of exercises in [AM69] and Section 1 of [B81].

4.1 Lemma. (a) $u(x)$ is invertible in $R[x]$ if and only if $u_{0}$ is invertible and the coefficient of each nonzero power of $x$ is nilpotent.

(b) $u(x)$ is invertible in $R[[x]]$ if and only if $u_{0}$ is invertible in $R$.

Note that if $e^{2} = e$ is an idempotent, then $(1 - 2e)^{2} = 1$, so:

4.2 Lemma. If $e$ is an idempotent, then $(1 - 2e)$ is a unit of $R$.

We combine these two lemmas to obtain:

4.3 Lemma. If $a(x)$ is an idempotent in $R[x]$ or $R[[x]]$, then $a(x) = a_{0} \in R$.

Proof: If $a(x) = \sum_{i=0}^{\infty} a_{i} x^{i}$ and $a(x) = (a(x))^{2}$, then $\sum_{i+j=n} a_{i} a_{j} = a_{n}$ for $n = 0, 1, 2, \ldots$. If $n = 0$, then $a_{0} = a_{0}^{2}$, so $(1 - 2a_{0})$ is a unit by the last lemma. Equating coefficients of $x$ yields $a_{1}(1 - 2a_{0}) = 0$, which implies that $a_{1} = 0$. Doing the same with the coefficients of $x^{2}$ yields $a_{2}(1 - 2a_{0}) = -a_{1}a_{1} = 0$, which implies that $a_{2} = 0$. Proceeding inductively, if $a_{1} = a_{2} = \cdots = a_{n-1} = 0$, then $a_{n}(1 - 2a_{0}) = -\sum_{i+j=n} a_{i} a_{j} = 0$. Thus $a_{n} = 0$ for each $n \geq 1$ and hence $a(x) = a_{0} \in R$. 

We now characterize von Neumann regular elements in $R[x]$ and $R[[x]]$. In the proof of the next theorem, we need the fact that if $a$ is a von Neumann regular element of a commutative ring, then there is unit $u$ such that $a^{2}u = a$, and hence that $au$ is an idempotent. See, for example [AHA04].

4.4 Theorem. Let $a(x) = \sum_{i=0}^{n} a_{i} x^{i}$. Then $a(x)$ is von Neumann regular in $R[x]$ if and only if $a(x)$ is a product of a von Neumann regular element in $R$ and a unit in $R[x]$.

Proof: If $a(x) \in \mathfrak{vr}(R[x])$, then there exists a unit $u(x) = \sum_{i=0}^{n} u_{i} x^{i} \in R[x]$ such that $a(x) = (a(x))^{2} u(x)$. Hence by Lemmas 4.1 and 4.3, we have

(iii) $a(x)u(x) = a_{0}u_{0} = (a_{0}u_{0})^{2}$ and

(iv) $\sum_{i+j=k} a_{i} u_{j} = 0$ for $k = 1, 2, 3, \ldots, n$.

By Lemma 4.1, $u_{j}$ is nilpotent if $j \geq 1$ and by the equation in (iv) for $k = 1$, $a_{1} = -u_{0}^{-1}a_{0}u_{1}$, which implies that $a_{1}$ is nilpotent. Similarly, $a_{2} = -u_{0}^{-1}(a_{0}u_{2} + a_{1}u_{1})$, which implies that $a_{2}$ is nilpotent. Proceeding inductively, if $a_{1}, a_{2}, \ldots, a_{n-1}$ are nilpotents, then $a_{n} = -u_{0}^{-1}\sum_{i+j=n} a_{i} u_{j}$. So $a_{k}$ is nilpotent
for each \( k \geq 1 \), while \( a_0 \in \text{vr}(R) \) and \( a(x) = a(x)a(x)u(x) = a(x)a_0u_0 \). Let 
\( v(x) = u_0 + a_1u_0^2x + a_2u_0^3x^2 + \cdots \) and note that it is a unit of \( R[x] \) by Lemma 4.1. Then:

\[
a(x) = \sum_{i=0}^{n} a_iu_0^ix^i = a_0^2u_0 + a_1a_0u_0x + a_2a_0u_0x^2 + \cdots \\
= a_0^2u_0 + a_1a_0a_0^2u_0^2x + a_2a_0a_0^2a_0^2x^2 + \cdots = a_0^3v(x)
\]

is the product of an element of \( \text{vr}(R) \) and a unit of \( R[x] \).

The converse is clear.

A similar argument will establish:

4.5 Theorem. If \( a(x) = \sum_{i=0}^{\infty} a_i x^i \), then \( a(x) \) is von Neumann regular in \( R[[x]] \) if and only if \( a(x) \) is a product of a von Neumann regular element in \( R \) and a unit in \( R[x] \).

By the last two theorems, \( xa(x) \in \text{vr}(R[[x]]) \) implies \( a(x) = 0 \), so we conclude this section with:

4.6 Corollary. For each ring \( R \), \( \mathfrak{M}(R[x]) = \{0\} \) and \( \mathfrak{M}(R[[x]]) = \{0\} \).

5. The ring \( C(X) \)

All topological spaces \( X \) are assumed to be Tychonoff spaces, \( \beta X \) the Stone-
Čech compactification of \( X \) and \( C(X) \) will denote the algebra of continuous real-
valued functions under the usual pointwise operations. For each \( f \in C(X) \), we
declare the zero set of \( f \) by \( Z(f) = \{ x \in X : f(x) = 0 \} \), and the cozero set \( \text{coz}(f) = X - Z(f) \). A point \( p \in X \) such that for every \( f \in C(X) \), \( f(p) = 0 \)
implies \( p \in \text{int} Z(f) \) is called a \( P \)-point, and \( X \) is called a \( P \)-space if each of its
points is a \( P \)-point. If \( x \in \beta X \), let \( M^x = \{ f \in C(X) : x \in \text{cl}_\beta X Z(f) \} \) and
\( O^x = \{ f \in C(X) : x \in \text{int}_\beta X \text{cl}_\beta X Z(f) \} \). The notation and terminology of
[13] is used. In this section we will characterize \( m \)-regular elements in \( C(X) \),
we will find for what spaces \( X \), \( \mathfrak{M}(C(X)) \) contains non zero elements.

Recall from Section 2 that \( R \) is a VNL ring if for each \( a \in R \), one of \( a \) or \( 1 - a \)
is von Neumann regular.

The next proposition is established in [Ahl04] and in [13].

5.1 Proposition. (a) \( C(X) \) is a VNR ring if and only if \( X \) is a \( P \)-space if and only if every \( G_\delta \)-set of \( X \) is open.

(b) \( C(X) \) is VNL ring if and only if at most one point of \( X \) is not a \( P \)-point (in which case \( X \) is said to be essentially a \( P \)-space).

The next simple lemma will be used below.
5.2 Lemma. If \( f \in \text{vr}(C(X)) \), then \( Z(f) \) is clopen.

**Proof:** As is noted just above Theorem 4.4, there is a unit \( u \) in \( C(X) \) such that \( f = f(fu) \) and \( fu \) is idempotent. Because the zerset of an idempotent is clopen, the conclusion follows. \qed

Thus we obtain:

5.3 Theorem. A function \( f \) is in \( \mathcal{M}(C(X)) \setminus \{0\} \) if and only if \( \text{coz}(f) \) is a nonempty clopen \( P \)-space.

**Proof:** Suppose that \( f \in \mathcal{M}(C(X)) \setminus \{0\} \), then \( f \in \text{vr}(C(X)) \) and so \( \text{coz}(f) \) is a nonempty clopen set by Lemma 5.2. Let \( G = \bigcap_{n=1}^{\infty} G_n \) be a \( G \)-set of \( X \) contained in \( \text{coz}(f) \) and suppose \( x \in G \). For each \( n \) there exists \( g_n \in C(X) \) such that \( g_n(x) = 0 \) and \( g_n(X \setminus G_n) = 1 \). Let \( g = \sum_{n=1}^{\infty} (g_n/2^n) \), then \( g \in C(X) \) and \( Z(g) = G \subseteq \text{coz}(f) \). Since \( f \in \text{vr}(C(X)) \), its zero set is clopen by Lemma 5.2. So, because \( Z(fg) = Z(f) \cup Z(g) \), \( Z(f) \cap Z(g) = \emptyset \), and \( Z(f) \) is clopen, it follows that \( Z(g) \) and hence \( \text{coz}(g) \) is clopen. Thus, by Proposition 5.1, \( \text{coz}(f) \) is a \( P \)-space.

Suppose conversely that \( \text{coz}(f) \) is a nonempty clopen \( P \)-space. Then \( C(X) \) is the direct product of \( C(\text{coz}(f)) \) and \( C(Z(f)) \), so \( f \in \mathcal{M}(C(X)) \setminus \{0\} \). \qed

5.4 Corollary. \( \mathcal{M}(C(X)) \neq \{0\} \) if and only if \( X \) contains a nonempty clopen \( P \)-space.

By making use of Theorem 1.3, we can describe \( \mathcal{M}(C(X)) \) more precisely.

If \( Y \) is a subset of \( X \), we let \( O^Y = \bigcap_{y \in Y} O^y \). Let \( P(X) \) be the set of all \( P \)-points in \( X \), then it is clear that \( O^{X-P(X)} = \bigcap_{y \notin P(X)} O^y \subseteq \text{vr}(C(X)) \) and so, \( O^{X-P(X)} \subseteq \mathcal{M}(C(X)) \). For each \( x \in \beta X \), \( mM^x = O^x \), using this together with Theorem 1.3 above we conclude that:

5.5 Corollary. \( \mathcal{M}(C(X)) = O^{X-P(X)} \) for any space \( X \).

We conclude with an interesting example.

5.6 Example. Let \( X_1 = (0, 1) \) with its usual topology and \( X_2 = \mathbb{N} \) with its discrete topology. Let \( X = X_1 \oplus X_2 \) and define \( f(x) = \begin{cases} 0 & \text{if } x \in X_1, \\ 1 & \text{if } x \in X_2 \end{cases} \), then \( f \in \mathcal{M}(C(X)) \setminus \{0\} \), while \( C(X) \) is not a VNR ring.

**References**


UNIVERSITY OF JORDAN, FACULTY OF SCIENCE, DEPARTMENT OF MATHEMATICS, AMMAN 11942, JORDAN

E-mail: eaboebsa@ju.edu.jo

HARVEY MUDD COLLEGE, CLAREMONT, CA 91711, U.S.A.

E-mail: henriksen@hmc.edu

UNIVERSITY OF JORDAN, FACULTY OF SCIENCE, DEPARTMENT OF MATHEMATICS, AMMAN 11942, JORDAN

E-mail: oalkam@ju.edu.jo

KENT STATE UNIVERSITY, KENT, OH 44242, U.S.A.

E-mail: fasmith@math.kent.edu

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