Ways in which $C(X)$ mod a Prime Ideal Can be a Valuation Domain; Something Old and Something New

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Abstract. $C(X)$ denotes the ring of continuous real-valued functions on a Tychonoff space $X$ and $P$ a prime ideal of $C(X)$. We summarize a lot of what is known about the residue class domains $C(X)/P$ and add many new results about this subject with an emphasis on determining when the ordered $C(X)/P$ is a valuation domain (i.e., when given two nonzero elements, one of them must divide the other). The interaction between the space $X$ and the prime ideal $P$ is of great importance in this study. We summarize first what is known when $P$ is a maximal ideal, and then what happens when $C(X)/P$ is a valuation domain for every prime ideal $P$ (in which case $X$ is called an SV-space and $C(X)$ an SV-ring). Two new generalizations are introduced and studied. The first is that of an almost SV-spaces in which each maximal ideal contains a minimal prime ideal $P$ such that $C(X)/P$ is a valuation domain. In the second, we assume that each real maximal ideal that fails to be minimal contains a nonmaximal prime ideal $P$ such that $C(X)/P$ is a valuation domain. Some of our results depend on whether or not $\beta\omega \setminus \omega$ contains a $P$-point. Some concluding remarks include unsolved problems.

1. Introduction

Throughout, $C(X)$ will denote the ring of real-valued continuous functions on a Tychonoff space $X$ with the usual pointwise ring and lattice operations and $C^*(X)$ will denote its subring of bounded functions, and all topological spaces considered are assumed to be Tychonoff spaces unless the contrary is stated explicitly. (Recall that $X$ is called a Tychonoff space if it is a subspace of a compact (Hausdorff) space. Equivalently if $X$ is a $T_1$ space and whenever $K$ is a closed subspace of $X$ not containing a point $x$, there is an $f \in C(X)$ such that $f(x) = 0$ and $f[K] = \{1\}$.) An element of $C(X)$ is nonnegative in the usual pointwise sense if and only if it is a square. So algebraic operations automatically preserve order. This makes the
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3. When every prime ideal of C(X) is a valuation prime;
SV rings and spaces

As in [DW96], a prime ideal P of C(X) is called a valuation prime if C(X)/P is a valuation domain. Chapter 14 of [GJ76] is devoted to the study of the set of prime ideals of C(X), and a little is said about the order-structure of C(X)/P when P is a prime ideal of C(X), but the first thorough study of valuation primes and their associated valuation domains appears in [CD86]. Inspired by this, a number of authors began to investigate rings C(X) and spaces X such that every prime ideal of C(X) is a valuation prime. Such rings and spaces are called SV-rings and SV-spaces respectively. See [HW92a], [HW92b], and [HLM94].

The order structure of C(X)/P is described completely when X is the one-point compactification W(ω+1) of W(ω) in [M90] and to a lesser extent in Chapter B of [DW96] when X is compact. See Section 6 below.

The Stone-Cech compactification βX is the compact space that contains X as a dense subspace such that each member of the subring C*(X) of bounded functions in C(X) has a (unique) extension in C(βX). Thus C*(X) and C(βX) are isomorphic.

What follows next will be used often below. See [GJ76].

1. Proposition and definitions. Suppose X is a Tychonoff space. Then:
(a) There is a bijection from βX onto the set M(C(X)) of maximal ideals of C(X) given by p → Mφ = {f ∈ C(X) : p ∈ clXZ(f)}.
(b) Each prime ideal of C(X) is contained in a unique maximal ideal Mφ, and the intersection of all the prime ideals contained in Mφ is OP = {f ∈ C(X) : clXZ(f) is a neighborhood of p}.
(c) The prime ideals containing a prime ideal P form a chain (that is, are totally ordered) under set inclusion, and P contains minimal prime ideals. The number of minimal prime ideals contained in the maximal ideal M is called its rank Rk(M) and is denoted by φ unless it is finite. If X is compact, the rank Rk(X) of X is the supremum of the ranks of the maximal ideals of C(X). It is known that if the rank of each maximal ideal of C(X) is finite, then Rk(X) < φ. (See [HLM94].)
(d) If p ∈ X, then Mφ is denoted by Mφp and OPβ is denoted by OP. If OPφ = Mφβ, then p is called a P-point. A space all of whose points are P-points is called a P-space. X is a P-space if and only if Z(f) is clopen for each f ∈ C(X) if and only if C(X) is a von Neumann regular ring. Moreover, every compact P-space is finite.
(e) If for all p ∈ βX, the prime ideals containing OPβ are totally ordered, then X is called an F-space. (Equivalently, X is an F-space if OPβ is prime for every p ∈ βX.) Every P-space is an F-space, every P-space is an SV-space and the implied inclusions are proper. Moreover, no SV-space can contain a nontrivial convergent sequence; that is, a closed copy of the one-point compactification W(ω+1) of the space W(ω) of finite ordinals. (See 3.4 below and [HLM94].)

If P ⊆ Q are prime ideals of C(X), then C(X)/Q is a homomorphic image of C(X)/P. Thus, if P is a valuation prime, then so is Q. Hence:

3.2 Proposition. X is an SV-space if and only if every minimal prime ideal of C(X) is a valuation prime.

Let mC(X) denote the set of minimal prime ideals of the ring C(X). If f ∈ C(X), let h(f) = {P ∈ mC(X) : f ∈ P} and let h*(f) = mC(X) \ h(f). Using (h*(f) : f ∈ C(X)) as a base for a topology, mC(X) becomes a zero-dimensional Hausdorff space called the space of minimal prime ideals of C(X). See [He85].

This space has been studied extensively, but we recall only those facts known about it that are relevant to this paper.

Recall that if I is an ideal of C(X) such that f ∈ I and Z(f) = Z(f) imply that f ∈ I, then I is called a z-ideal. As is shown in [He85], the map P → P ∩ C*(X) is an order preserving homeomorphism of mC(X) onto mC*(X). So C(X) and C(βX) have homeomorphic spaces of minimal prime ideals. Again a prime z-ideal P of C(X) is valuation prime if and only if P ∩ C*(X) is a valuation prime of C*(X) by Corollary 2.1.12 of [CD86]. Thus we have:

3.3 Proposition. X is an SV-space if and only if βX is an SV-space.

Thus, to determine the algebraic properties of an SV-ring C(X), there is no loss of generality in assuming that X is compact.

3.4 Remark. It has long been known that every F-space is an SV-space. (See, for example [L86].) If X and Y are two disjoint F-spaces, then the attaching of X and Y at two non F-points respectively of X and Y serves as an example of an SV-space which is not an F-space and consequently the class of SV-spaces contains the class of F-spaces properly. (This is noted in [HW92] and [HW92b], and implicitly in [CD86].)

Recall that a space X is said to be C*-embedded (resp. C-embedded) in a space Y if the map f → f|Y is a surjection of C*(X) onto C*(Y) (resp. of C(X) onto C(Y)). We close this section with more known facts about SV-spaces.

3.5 Proposition and remarks
(a) Every C*-embedded subspace of an SV-space is an SV-space and consequently closed subspaces of compact SV-spaces are SV-spaces. (See [HW92a].)
(b) If a compact space X can be expressed as the union of finitely many closed subspaces such that each of them is an SV-space, then X becomes an SV-space, but not every SV-space can be represented in this way. (See [HW92a] and [L83]. The example in [L83] is the result of a complicated construction.)
(c) If X is compact and each point of X has a closed neighborhood that is an F-space, then X becomes an SV-space. Though the converse need not hold. Let X be the union of two disjoint copies of βω (ω denotes the set of countable ordinals) and Y the space by identifying the corresponding points of
4. The first generalization; almost SV rings and spaces

Below $\beta \omega$ and $\omega \omega$ will abbreviate $\beta(W(\omega))$ and $W(\omega + 1)$ respectively.

4.1 Definition. $C(X)$ is called an almost SV-ring and $X$ an almost SV-space if each maximal ideal of $C(X)$ contains a valuation prime that is a minimal prime ideal.

Next, it is shown how to create a large class of almost SV-spaces that are not SV-spaces.

4.2 Theorem. The space obtained by attaching the nonisolated point of $W(\omega + 1)$ to a compact $F$-space at a non $F$-point is an almost SV-space that is not an SV-space.

Proof. Let $Y$ denote a compact $F$-space with a point $q$ such that $O_q$ is not maximal. (For example, $Y$ could be $\beta \omega$ and $q$ any point of its nonisolated points.) Let $X$ denote the result of attaching the spaces $Y$ and $\omega q$ at the points $q \in Y$ and $q \in \omega q$, and call the resulting point $p$. By 3.1(e), $X$ is not an SV-space because it contains a sequence of distinct points converging to $p$. Because $Y$ is an $F$-space and each point of $\omega q$ other that $\omega q$ is a $P$-point, to see that $X$ is an almost SV-space, it suffices to show that there is a minimal valuation prime of $C(X)$ containing $O_p$.

Let $\varphi$ and $\psi$ denote respectively the restriction maps of $C(X)$ onto $C(\omega q)$ and $C(Y)$. Clearly each is a surjective homomorphism. Because $O_q$ is not maximal, its inverse image $\psi^{-1}(O_q)$ is a nonmaximal prime ideal of $C(X)$ containing $O_p$. We show next that $O_p = \varphi^{-1}(O_q) \cap \psi^{-1}(O_q)$. For, $f$ is in this intersection if and only if its restriction to $\omega q$ vanishes on a neighborhood of $\omega q$, and its restriction to $Y$ vanishes on a neighborhood of $q$ if and only if $f$ vanishes on a neighborhood of $p$.

Because in any commutative ring a prime ideal that contains the intersection of two ideals must contain one of them, it follows that any prime ideal of $C(X)$ that contains $O_p$ must contain at least one of $\varphi^{-1}(O_q)$ or $\psi^{-1}(O_q)$. Moreover, because $q$ is not a $P$-point of $Y$, $\varphi^{-1}(O_q)$ is not contained in $\psi^{-1}(O_q)$. Hence there cannot be any prime ideal of $C(X)$ containing $O_p$ and properly contained in $\psi^{-1}(O_q)$. Thus the latter is a minimal prime ideal of $C(X)$ containing $O_p$.

Suppose $\pi : C(Y) \to C(Y)/O_q$ is the canonical homomorphism. Then $\pi \circ \psi : C(X) \to C(Y)/O_q$ is a surjective homomorphism with whose kernel $\ker(\pi \circ \psi) = \psi^{-1}(O_q)$. So $C(X)/\ker(\pi \circ \psi)$ and $C(Y)/O_q$ are isomorphic. Because $Y$ is an $F$-space, this latter is a valuation domain and we may conclude that $\psi^{-1}(O_q)$ is a minimal valuation prime ideal. This concludes the proof of the theorem.

We digress to quote some results in [CD86] that will be useful in what follows.

4.3 Concepts and results from [CD86].

(a) Suppose $p$ is a point in a Tychonoff space $X$ and $\mathcal{I}$ is a $Z$-filter of subsets of $X$. If for every $f \in C(X \setminus \{p\})$ such that $0 \leq f \leq 1$, there is $Y \in \mathcal{I}$ such that $\lim_{Y \to Y} f|Y$ exists, then $\mathcal{I}$ is called a $P(p)$ filter. See [CD86] for a discussion of the properties of such filters. The authors do not describe $P(p)$ filters as $Z$-filters, but treat them as one whenever they use it.

(b) Suppose $X$ is compact and each of its points is a $G_\delta$ point. Substantial use will be made below of a mapping $\gamma$ introduced in a more general setting by C. Kohls in [K58] and described in more detail in [CD86]. See Section 2.2 of [CD86] for proofs and more details about the assertions made below.

1. Suppose $p \in X$ is nonisolated and $Y = X \setminus \{p\}$. Then $\gamma$ is a bijection from the set $Q$ of prime $Z$-ideals of $C(Y)$ such that $\mathcal{Z}[Q]$ converges to some point in $\beta Y \setminus \{p\}$ onto the family $\mathcal{I}$ of all nonmaximal prime $Z$-ideals of $C(X)$ contained in $M_p$. Then $\gamma(Q)$ is defined by $\mathcal{Z}[\gamma(Q)] = \{c_{\gamma}X : E \in \mathcal{Z}[Q]\} = \{Y \cup \{p\} : Y \in \mathcal{Z}[Q]\}$.

2. The prime $Z$-ideal $Q$ is maximal if and only if $\gamma(Q)$ is an immediate prime $Z$-ideal of $M_p$ by 3.2(2) of [CD86].

3. $\mathcal{Z}[Q]$ is a $P(p)$ filter if and only if $\mathcal{Z}[\gamma(Q)]$ is a $P(p)$ filter.

4. If $P$ is a valuation prime contained properly in $M_p$ and $Q \subset P$ is a minimal prime ideal, then $\mathcal{Z}[Q]$ is a $P(p)$ filter. (See Theorem 2.2.2 of [CD86].)

Recall that $vX = \{p \in \beta X : M^p$ is a real maximal ideal$\}$ and that $C(vX)$ and $C(X)$ are isomorphic. $vX$ is called the (Hewitt) realcompactification of $X$. The proof of the next result is an exercise.

4.4 Theorem. The following assertions are equivalent.

1. $X$ is an almost SV-space.
2. $vX$ is an almost SV-space.
3. $\beta X$ is an almost SV-space.

4.5 Corollary. If $Y$ is a dense $C^*$-embedded subspace of an almost SV-space $X$, then $Y$ is an almost SV-space.

Proof. For then $\beta Y$ and $\beta X$ are homeomorphic, so the conclusion follows from Theorem 4.4.

Recall again that the question of whether a compact space of finite rank is an SV-space was left as an open problem in Section 6 of [HLMW94]. So, the second
part of the hypothesis of the next theorem may be redundant, and its conclusion may be too weak.

4.6 Theorem. If X is a compact space of finite rank and each of its points has a compact neighborhood that is an almost SV-space, then X is an almost SV-space.

Proof. Because X is compact, \( \{ M_p : p \in X \} \) is the set of all maximal ideals of C(X). We need to show that each \( M_p \) contains a minimal prime ideal that is a valuation prime. For each \( p \in X \), there is a compact neighborhood \( T = T(p) \) of \( p \) that is an almost SV-space. Let \( \varphi : C(X) \to C(T) \) denote the map that sends each \( f \in C(X) \) to its restriction to \( T \). The map \( \varphi \) is an epimorphism since \( T \) is C-embedded in \( X \). As usual, \( O_p = \{ f \in C(X) : p \in \text{int} \mathcal{X} \mathcal{Z}(f) \} \) and letting \( O_p^\varphi = \{ f \in C(T) : p \in \text{int} \mathcal{X} \mathcal{Z}(f) \} \), we see that \( \varphi^{-1}(O_p) = O_p \). Note also that the inverse image under \( \varphi \) of a pair of incomparable prime ideals of C(T) containing \( O_p^\varphi \) is a pair of incomparable prime ideals of C(X) containing \( O_p \) because \( \varphi \) is an epimorphism.

Let \( m \) and \( n \) denote respectively the ranks of \( p \) with respect to \( T \) and \( X \). Because \( X \) is compact, it follows from Corollary 1.8.2 of [HLMW94] that \( m \leq n \). It follows that \( O_p^\varphi \) is the intersection of \( m \) incomparable minimal prime ideals \( \{ P_i^\varphi \}_{i=1}^m \) of \( C(T) \) and hence that \( O_p \) is the intersection of the minimal prime ideals \( \{ \varphi^{-1}(P_i) \}_{i=1}^m \) in \( C(X) \). In any commutative ring, if a prime ideal contains a finite intersection of ideals, it must contain one of them. So, we can conclude that \( \varphi^{-1}(P_i) \) is the collection of all minimal prime ideals of \( C(X) \) that contain \( O_p \). Because \( T \) is an almost SV-space, there is a \( j \) such that \( 1 \leq j \leq m \) and \( P_j^\varphi \) is a maximal valuation prime of \( C(T) \) containing \( O_p^\varphi \). If \( \pi : C(T) \rightarrow C(T)/P_j^\varphi \) denotes the canonical homomorphism, then \( \pi \circ \varphi \) is an epimorphism from \( C(T)/P_j^\varphi \) onto \( C(X)/\varphi^{-1}(P_j) \) whose kernel is \( \varphi^{-1}(P_j) \). Therefore \( C(X)/\varphi^{-1}(P_j) \) is a maximal valuation prime of \( C(X) \) contained in \( M_p \). Since \( p \in X \) is arbitrary, this completes the proof.

Use will be made below of the concept that follows in deriving a sufficient condition for a compact perfectly normal space to be an almost SV-space.

4.7 Definition. For any space \( X \), a point \( p \in \beta X \) such that \( O^p \) is a prime ideal of \( C(X) \) is called a \( \beta F \)-point. If \( O^p \) is a valuation prime, then \( p \) is called a special \( \beta F \)-point.

No example is known of a \( \beta F \)-point that is not a special \( \beta F \)-point.

4.8 Lemma. Suppose \( p \) is a nonisolated \( G_\delta \)-point of a compact space \( X \), and let \( Q \) denote a prime \( z \)-ideal of \( C(X \setminus \{ p \}) \) such that the prime \( z \)-filter \( Z[Q] \) converges to some point in \( \beta(\mathcal{X} \mathcal{Z}(Q)) \setminus \{ p \} \). Then the following are equivalent:

1. For every \( f \in C(X \setminus \{ p \}) \) such that \( 0 \leq f \leq 1 \), there exists a \( Y \in Z[Q] \) and \( g \in C(X) \) such that \( f Y = g Y \).
2. For every \( f \in C(X \setminus \{ p \}) \) such that \( 0 \leq f \leq 1 \), there exists a \( Y \in Z[Q] \) such that \( \lim_{x \to p} f Y \) exists, i.e., \( Z[Q] \) is a \( P(p) \) filter.

Proof. If (1) holds, then \( \lim_{x \to p} f Y \) exists and consequently \( \lim_{x \to p} f Y \) also exists. So (2) holds.

If (2) holds, define \( h : Y \cup \{ p \} \to \mathbb{R} \) by letting \( h|Y = f|Y \) and \( h(p) = \lim_{x \to p} f Y \). Clearly \( h \in \mathcal{C}(Y \cup \{ p \}) \). Since \( X \) is compact and \( Y \cup \{ p \} \) is a closed subset of \( X \), and hence is \( C \)-embedded in \( X \). If \( g \) is an extension of \( h \) over \( X \), then \( g|Y = f|Y \).

Combining Lemma 4.8 and Theorem 2.3.2 of [CD86], we obtain:

4.9 Theorem. Suppose \( p \) is a nonisolated \( G_\delta \)-point of a compact space \( X \), and let \( Q \) denote a prime \( z \)-ideal of \( C(X \setminus \{ p \}) \) such that the prime \( z \)-filter \( Z[Q] \) converges to some point in \( \beta(\mathcal{X} \mathcal{Z}(Q)) \setminus \{ p \} \). Then the following are equivalent:

1. \( \gamma(Q) \) is a valuation prime \( z \)-ideal of \( C(X) \) contained in \( M_p \).
2. \( Q \) is a valuation prime \( z \)-ideal and \( Z[Q] \) is a \( P(p) \) filter.

4.10 Theorem. If \( X \) is compact and perfectly normal, and for every nonisolated point \( p \) of \( X \), there is a free \( P(p) \)-ultrafilter \( Z[M^p] \) on \( X \setminus \{ p \} \) such that \( q \) is a special \( \beta F \)-point of \( \mathcal{B}(X \setminus \{ p \}) \), then \( X \) is an almost SV-space.

Proof. A nonisolated point \( p \) of the perfectly normal space \( X \) is a \( G_\delta \)-point. Suppose \( Z[M^p] \) is a free \( P(p) \)-ultrafilter on \( X \setminus \{ p \} \) such that \( q \) is a special \( \beta F \)-point. Thus \( O^q \) becomes a valuation prime ideal of \( C(X \setminus \{ p \}) \) and \( Z[O^q] \) clearly converges to \( q \) in \( \beta(\mathcal{X} \mathcal{Z}(Q)) \setminus \{ p \} \). Since \( \gamma \) is a bijection, \( \gamma(O^q) \) is a minimal prime ideal of \( C(X) \) contained in \( M_p \). Because \( Z[M^p] \) is a \( P(p) \)-filter, \( \gamma(M^p) \) becomes an immediate prime \( z \)-ideal of \( M_p \) which is a valuation prime by 2.3.3 of [CD86]. Now \( \gamma(O^q) \) is a minimal prime ideal of \( C(X) \) contained in \( \gamma(M^p) \), which is properly contained in \( M_p \) and since \( \gamma(M^p) \) is a valuation prime, this implies that \( \gamma(O^q) \) is a \( P(p) \)-filter by 2.2.2 of [CD86]. Consequently, by 4.3(h)(3) above, \( Z[O^q] \) becomes a \( P(p) \)-filter. Finally, because \( O^q \) is a valuation prime and \( Z[O^q] \) is a \( P(p) \)-filter, \( \gamma(O^q) \) becomes a valuation prime by Theorem 4.9. But \( p \) is an arbitrary nonisolated point, so this completes the proof.

4.11 Theorem. \( \omega \) is an almost SV-space if and only if there exists a free ultrafilter \( \Psi \) on \( \omega \) such that for every \( f \geq 0 \) in \( C^{\ast}(\omega) \), there is a \( Y \in \Psi \) such that \( \lim_{x \to \omega} f|Y \) exists.

Proof. Let \( \varphi \) be the unique continuous extension of the inclusion map \( i : \omega \to \omega \) over \( \beta \omega \). Recall from 4.3(b)(1) that the map \( \varphi \) is a bijection of the family of all prime \( z \)-ideals \( Q \) of \( C(\omega) \) such that \( Z[Q] \) converges to point of \( \varphi^{-1}(\omega) \) (where \( \varphi^{-1}(\omega) = \beta \omega \setminus \omega \)) onto the family of all nonmaximal prime \( z \)-ideals of \( C(\omega) \) contained in \( M_\omega \). Since \( \omega \) is a \( P \)-space, the set of prime \( z \)-ideals \( Q \) of \( C(\omega) \) such that \( Z[Q] \) converges to point of \( \beta \omega \setminus \omega \) is the set of maximal ideals \( M^\omega \) such that \( q \in \beta \omega \setminus \omega \). So by 4.3(h)(1), the set of minimal prime ideals of \( C(\omega) \) contained in \( M_\omega \) is given by \( \{ \gamma(M^\omega) : q \in \beta \omega \setminus \omega \} \). Thus \( \omega \) is an almost SV-space if and only if \( \gamma(M^\omega) \) is a valuation prime for some \( q \in \beta \omega \setminus \omega \) if and only if \( Z[M^\omega] \) is a \( P(\omega) \)-ultrafilter on \( \omega \) (by Corollary 3 of [CD86]) if and only if there exists a free
ultrafilter \( \Psi \) on \( \omega \) such that for every \( f \geq 0 \) in \( C^*(\omega) \), there exists \( Y \in \Psi \) such that \( \lim_{f \to \omega} f|Y \) exists.

Henceforth, we will write \( A \cong B \) to abbreviate the statement that the rings \( A \) and \( B \) are isomorphic, and we will write \( X \simeq Y \) to abbreviate the statement that the topological spaces \( X \) and \( Y \) are homeomorphic.

The Stone extension theorem states that if \( f : X \to Y \) is continuous and \( Y \) is compact, then \( f \) has a continuous extension \( f^* : \beta X \to Y \). (See 6.5 of [GJ76].)

An algebra \( A \) such that \( C^*(X) \subset A \subset C(X) \) is said to be an intermediate algebra of \( C(X) \) and is said to be an \( \alpha \)-type algebra. If also \( A \cong C(Y) \) for some Tychonoff space \( Y \). We let \( A^* = \{ f \in A : |f| \) is bounded by some positive integral multiple of 1 \}. If \( A \) is an intermediate algebra of \( C(X) \) then clearly \( A^* = C^*(X) \). For more background on intermediate \( \alpha \)-type algebras, see [DG9].

**4.12 Theorem.** \( C(X) \) is an almost \( SV \)-ring if and only if every \( \alpha \)-type intermediate algebra of \( C(X) \) is an almost \( SV \)-ring.

**Proof.** Let \( A \) be an intermediate \( \alpha \)-type algebra of \( C(X) \) and \( \nu_A X = \{ p \in \beta X : f^*(p) \in \mathbb{R}, \text{ for all } f \in A \} \) where \( f^* \) is the Stone extension of \( f \) to \( \beta X \) to the two-point compactification \( \mathbb{R} \cup \{ \pm \infty \} \) of \( \mathbb{R} \). Clearly \( X \subset \nu_A X \subset \beta X \). By Corollary 2.3 of [Hjo61], \( \text{Max}(A) \approx \text{Max}(A^*) \) and \( \text{Max}(C^*(X)) \approx \beta X \). Therefore \( \text{Max}(A) \approx \beta X \approx \beta(\nu_A X) \). Consequently \( A \cong C(\nu_A X) \) and the rest of the proof follows by Theorem 4.4.

5. Product spaces and set-theoretic considerations

In Theorem 3.1 of [CD86], it is shown that if \( \omega \omega \) contains a nonmaximal valuation prime, then the space \( \omega \omega \setminus \omega \) contains a \( P \)-point. It is noted also in this paper that W. Rudin showed that if CH holds, then \( \omega \omega \setminus \omega \) contains a dense set of \( P \)-points, and Shelah showed that there are models of ZFC in which \( \omega \omega \setminus \omega \) has no \( P \)-points. (See [Wi82].) It follows that there are models of ZFC in which \( \omega \omega \) is not an almost \( SV \)-space. In Theorem 3.3.4 of that paper, it is shown that these results also hold if \( \omega \) is replaced by any infinite discrete space.

This yields another difference between \( SV \) and almost \( SV \)-spaces. While closed subspaces of compact \( SV \)-spaces are \( SV \)-spaces, there are models of ZFC in which the corresponding result for almost \( SV \)-spaces need not hold. In particular, by Theorem 4.2, in a model in which \( \omega \omega \setminus \omega \) has no \( P \)-points, the space obtained by attaching a copy of \( W(\omega + 1) \) to a point of \( \omega \omega \setminus \omega \) is an almost \( SV \)-space with a countable closed subspace that is not an almost \( SV \)-space.

Observe that a space is the free union \( X_1 \cup X_2 \) of spaces \( X_1 \) and \( X_2 \) if and only if \( C(X) \) is the direct sum \( C(X_1) \oplus C(X_2) \). Because every maximal [resp. proper prime] ideal of \( C(X_1 \cup X_2) \) is the direct sum of a maximal [resp. proper prime] ideal in one coordinate and the whole ring in the other, it follows that \( C(X_1 \cup X_2) \) is an almost \( SV \)-space if and only if \( C(X_i) \) is an almost \( SV \)-space for \( i = 1, 2 \).

Recall also from [HMW03] that \( X \) is called a quasi \( P \)-space if each of the prime \( z \)-ideals of \( C(X) \) is minimal or maximal. The following facts will be used below.

**Fact 1.** The one-point compactification \( \alpha D \) of an infinite discrete space \( D \) is a quasi \( P \)-space. (See 2.4 of [HMW03].)

**Fact 2.** Every infinite locally compact quasi \( P \)-space \( T \) is a free union of one-point compactifications of infinite discrete spaces. This free union is finite if and only if \( T \) is compact. (See 4.1 and 6.1 of [HMW03].)

**Fact 3.** A prime ideal \( P \) of \( C(X) \) is minimal if and only if \( f \notin P \) implies there is a \( g \notin P \) such that \( fg = 0 \). The space \( mC(X) \) of minimal prime ideals of \( C(X) \) (with the topology described just after Prop. 3.2) is always a countably compact zero-dimensional Hausdorff space. Moreover, \( mC(X) \) is compact if and only if whenever each function in a prime ideal \( Q \) of \( C(X) \) has nonempty interior, it follows that \( Q \in mC(X) \). (See [HJ65].)

A space \( X \) such that whenever \( V \subset \omega X \), there is a \( W \subset \omega X \) such that \( V \cap W = \emptyset \) and \( V \cup W \) is dense in \( X \), is said to be \( \omega \)-complemented. It is well known that \( mC(X) \) is compact if and only if \( X \) is \( \omega \)-complemented. See [HW04]. It will be noted in 5.3 below that if \( D \) is an uncountable discrete space, then \( \omega \omega \) implies that \( \alpha D \) is an almost \( SV \)-space, even though \( \omega \omega \) is \( \omega \)-complemented while \( \alpha D \) is not \( \omega \)-complemented.

5.1 Theorem.

(a) If \( \omega \omega \) is an almost \( SV \)-space and \( Y \) is a compact metrizable almost \( SV \)-space with a dense set of isolated points, then \( \omega \omega \times Y \) is an almost \( SV \)-space.

(b) If \( \omega \omega \) is not an almost \( SV \)-space, then neither is \( \omega \omega \times Y \).

**Proof.** (a) It follows easily from Fact 3 and the observation that \( \omega \omega \times Y \) is compact if it suffices to show for any \( y \in Y \) that \( \text{Max}(\omega \omega \times Y) \) contains a valuation prime that is a minimal prime ideal. Using Fact 3 again yields this result if \( y \) is an isolated point of \( Y \), so we may assume it is not isolated. Because \( \omega \omega \times Y \) is metrizable and \( Y \) has a dense set of isolated points, there is a sequence \( \{ x_n \} \) of isolated points of \( \omega \omega \times Y \) that converges to \( (\omega, y) \). Clearly this sequence together with \( (\omega, y) \) is a subspace \( T \) of \( \omega \omega \times Y \) homeomorphic to \( \omega ) \). Because this latter is an almost \( SV \)-space, there is a valuation prime \( P \) that is a minimal prime ideal of \( C(T) \) contained in the maximal ideal \( \{ f \in C(T) : f(\omega, y) = 0 \} \) of \( C(T) \). Let \( \rho \) denote the map that sends \( f \in \text{Max}(\omega \omega \times Y) \) to its restriction to \( T \), and let \( \varphi \) be a mapping that sends each member of \( C(T) \) to its coset mod \( P \) in the valuation domain \( D = C(T)/P \). Clearly \( \text{ker}(\rho \circ \varphi) = \rho^{-1}(P) \) is a valuation prime ideal contained in \( \text{Max}(\omega \omega \times Y) \). It remains only to prove that it is a minimal prime ideal.

Because \( \omega \omega \times Y \) is metrizable, the space of minimal prime ideals of \( C(\omega \omega \times Y) \) is compact. So \( \rho^{-1}(P) \) is minimal provided that each of its elements is a zero divisor. (See [HJ65]). That will be the case if each \( f \in P \) has a zero set with nonempty interior. Now \( f \in \rho^{-1}(P) \) implies \( \rho(f) \in P \) implies \( f|T \in P \) implies \( \text{int}Z(f) \neq \emptyset \) because \( P \) is a minimal prime ideal of \( C(T) \). Also, because \( \{ x_n \} \) is a sequence of isolated points of \( \omega \omega \times Y \), it follows that \( \text{int}Z(f) \neq \emptyset \). So (a) holds.
(b) If \( p \) is an isolated point of \( Y \), then \( \omega \times Y \) is a free union of \( X_1 = \omega \times \{ p \} \) and \( X_2 = \omega \times (Y \setminus \{ p \}) \). Therefore \( C(\omega \times Y) \) is an almost \( SV \)-ring if and only if each \( C(X_i) \) is for \( i = 1, 2 \). Since \( X_1 \approx \omega \), it follows that \( \omega \) is an almost \( SV \)-space.

With the aid of a routine induction, (b) implies:

5.2 Corollary. \( \omega \) is an almost \( SV \)-space if and only if \( (\omega \omega)^n \) is an almost \( SV \)-space for any positive integer \( n \).

The proof of the next theorem depends on some results in [HMW03] where it is shown that a compact space is a quasi \( P \)-space if and only if it is a finite free union of one-point compactifications of discrete spaces.

5.3 Theorem. Let \( D \) be any uncountable discrete space and let \( \alpha D \) denote its one-point compactification. If \( \omega \) is an almost \( SV \)-space then so is \( \alpha D \).

Proof. By the preceding remarks, \( \alpha D \) is a quasi \( P \)-space and hence every prime \( z \)-ideal of \( C(\alpha D) \) is either minimal or maximal. Clearly \( \alpha D \) contains a copy of \( \omega \); say \( T \). Since \( \omega \) is an almost \( SV \)-space, there is a minimal nonmaximal prime ideal \( P \) of \( C(T) \) which is a valuation prime. Let \( \varphi \) denote the restriction mapping of \( C(\alpha D) \) onto \( C(T) \).

\( P \) is a \( z \)-ideal since it is a minimal nonmaximal prime ideal of \( C(T) \). Therefore \( \varphi^{-1}(P) \) is a prime \( z \)-ideal of \( C(\alpha D) \) which cannot be maximal. So, because \( \alpha D \) is a quasi \( P \)-space, \( \varphi^{-1}(P) \) is minimal. Since \( C(\alpha D) / \varphi^{-1}(P) \) and \( C(T)/P \) are isomorphic, this completes the proof.

Next, we prove two results on product spaces under the assumption that \( \omega \) is an almost \( SV \)-space.

5.4 Lemma. If \( \alpha S = S \cup \{ s \} \) and \( \alpha T = T \cup \{ t \} \) are the one-point compactifications of infinite discrete spaces \( S \) and \( T \), and \( \omega \) is an almost \( SV \)-space, then \( X = \alpha S \times \alpha T \) is an almost \( SV \)-space.

Proof. It is shown in Theorem 5.3 that if \( \omega \) is an almost \( SV \)-space, then the one-point compactification of any infinite discrete space is an almost \( SV \)-space. So, we need only show that \( M_{(s,t)} \) contains a valuation prime ideal that is minimal. It is easy to find a sequence \( \{ x(n) \} \) of distinct isolated points of \( X \) that converges to \( (s, t) \). If \( \{ x(n) \} \) and \( \{ t(n) \} \) are sequences of distinct isolated points of \( S \) and \( T \), let \( x(n) = (s(n), t(n)) \). If \( Y = \{ x(n) \} \cup \{(s, t)\} \), then \( Y \approx \omega \). So there is a minimal prime valuation prime ideal \( P \) of \( C(Y) \) contained in \( \{ f \in C(Y) : f(p, q) = 0 \} \). If \( \varphi \colon C(X) \to C(Y) \) is the restriction map, then clearly \( \varphi^{-1}(P) \) is valuation prime.

So, we need only show that the prime ideal \( \varphi^{-1}(P) \) is minimal. By Fact 3, we need only show that if \( f \not\in \varphi^{-1}(P) \), there is \( g \not\in \varphi^{-1}(P) \) such that \( fg = 0 \). Note also that the zero set of an element of a minimal prime ideal of \( C(Y) \) is infinite.

Suppose first that \( (s, t) \in \text{int}_{Y} Z(f(Y)) \), in which case \( \text{coz}(f(Y)) \) is a finite set of isolated points of \( Y \). If \( Z(f(Y)) \setminus (s, t) = \{ x(n) \}_{n=1}^{\infty} \), let \( g(Y)(p(n)) = \frac{1}{n} \) for \( n \geq 1 \), and \( g(Y) = 0 \) otherwise. Then \( f(Y)(g(Y)) = 0 \), while \( g(Y) \not\in P \) because its zero set is finite. It follows that \( \varphi^{-1}(P) \) is minimal.

5.5 Theorem. If \( \omega \) is an almost \( SV \)-space, then the product of two infinite compact quasi \( P \)-spaces \( X \) and \( Y \) is an almost \( SV \)-space that is not a quasi \( P \)-space (or an \( SV \)-space).

Proof. By Fact 2, both \( X \) and \( Y \) are free unions of finitely many one point compactifications of infinite discrete spaces, so \( X \times Y \) is a finite free union of spaces of the form \( \alpha S_i \times \alpha T_j \) for infinite discrete spaces \( S_i \) and \( T_j \). Each of these summands is an almost \( SV \)-space by the lemma, as is their free union.

If \( X \times Y \) were a quasi \( P \)-space, then so would each of \( \alpha S_i \times \alpha T_j \). By 5.1 of [HMW03], this cannot be the case since each of the latter factors are compact and infinite. Finally, because \( X \times Y \) contains a convergent sequence, it cannot be an \( SV \)-space.

6. Some consequences of the assumption that \( \alpha \omega \) is an almost \( SV \)-space

Henceforth the assumption that the one-point compactification of the countable discrete space \( \omega \) is an almost \( SV \)-space will be denoted by \( \Omega_{\omega} \). This assumption has been used since the beginning of Section 5.

6.1 Theorem. The following assertions are equivalent:

(a) \( \Omega_{\omega} \) holds.

(b) There is a \( p \in \beta \omega \setminus \omega \) such that the maximal ideal \( M_p \) of \( C(\beta \omega) \) is the immediate prime \( z \)-ideal successor of \( D_p \) in the class of all \( z \)-ideals of \( C(\beta \omega) \).

Proof. Let \( i \) denote the restriction mapping from \( C(\beta \omega) \) to \( C(\omega) \). If \( p \in \beta \omega \setminus \omega \), then since \( \omega \) is a discrete space, \( i^{-1}(M_p) = O_p \subset C(\beta \omega) \). Because \( \beta \omega \setminus \omega \) is a zeroregion, the prime \( z \)-filter \( Z(i^{-1}(M_p)) \) on \( \beta \omega \) has an immediate successor in the class of \( z \)-filters which is exactly the \( z \)-filter generated by \( Z(i^{-1}(M_p)) \) together with \( \beta \omega \setminus \omega \) by Theorem 3.5 of [GJ60]. It follows from Theorem 3.10 of the same paper that this successor is \( Z(M_p) \) if and only if \( p \) is a \( P \)-point of \( \beta \omega \setminus \omega \), which as noted above is equivalent to \( \Omega_{\omega} \).

6.2 Lemma. Suppose:

(a) \( \Omega_{\omega} \) holds.

(b) \( X \) and \( mC(X) \) are compact (e.g., if \( X \) is compact and perfectly normal), and

(c) for each nonisolated point \( p \in X \), there exists an infinite set of isolated points \( D_p \) of \( X \) such that \( D_p \cup \{ p \} \) and the point compactification \( \alpha D_p \) of \( D_p \) are homeomorphic.

Then \( X \) is an almost \( SV \)-space.

Proof. Since (a) holds, \( Y = \alpha D_p \) is an almost \( SV \)-space as noted in the first paragraph of this section. So there is a \( P \in mY \) that is a valuation prime contained in \( \{ f \in C(Y) : f(p) = 0 \} \). Letting \( \varphi : C(X) \to C(Y) \) denote the restriction map, we see that \( C(X)/\varphi^{-1}(P) \cong C(Y)/P \). It remains only to show that \( \varphi^{-1}(P) \in mX \).

Now \( g \in \varphi^{-1}(P) \) implies \( g \in P \). By (c), \( \text{int}_{Y} Z(g) \neq \emptyset \) since \( P \in mC(Y) \). So
since each point of \( \text{int}_Y Z(g) \) is isolated in \( X \), we know that \( \text{int}_X Z(g) \) is nonempty as well. From Fact 3 of Section 5, we conclude that \( \varphi^{-1}(P) \in mC(X) \). 

A topological space \( X \) is said to be scattered or dispersed if each nonempty subspace \( Y \) contains an isolated point of \( Y \). A compact scattered space is necessarily zero-dimensional.

If \( X \) is a space, let \( X^{(0)} = X \), \( X^{(1)} = X \setminus \text{Is}(X) \), and for any ordinal \( \eta \), let \( X^{(\eta+1)} = (X^{(\eta)})^{(1)} \). If \( \eta \) is a limit ordinal, then \( X^{(\eta)} \) denotes intersection of all \( X^{(\beta)} \) such that \( \beta < \eta \). From cardinality considerations there is an ordinal \( \alpha \) such that \( X^{(\alpha)} = X^{(\beta)} \), for each \( \beta > \alpha \). If there is an \( \alpha \) such that \( X^{(\alpha)} = \varnothing \), then \( X \) is scattered and the least such \( \alpha \) is called the \( CB \)-index of the scattered space \( X \).

These notions abound in general topology. See, for example, [LR81] or [Sn71].

6.3 Theorem. If \( \Omega_{SV} \) holds, then every compact metrizable scattered space \( X \) of Cantor-Bendixon index \( \leq 3 \) is an almost \( SV \)-space.

\textbf{Proof.} The hypothesis of Lemma 6.2 will be verified. If \( CB(X) = 1 \), then \( X \) is finite and hence is a \( P \)-space. If \( CB(X) = 2 \), then the set of nonisolated points of \( X \) is finite and therefore for every nonisolated point there exists a sequence of isolated points converging to it. If \( CB(X) = 3 \), then since \( X \) is compact and metrizable, \( X^{(2)} \) is finite and therefore for all but finitely many points of \( X^{(1)} \), there is a sequence of isolated points of \( X \) converging to the point. Let \( p \in X^{(1)} \) and \( \{x_n\}_{n=1}^{\infty} \) be a sequence of nonisolated points converging to \( p \); i.e., \( \{x_n\}_{n=1}^{\infty} \subseteq X^{(1)} \). By our earlier assertion, for all but finitely many members of \( \{x_n\}_{n=1}^{\infty} \), there is a sequence of isolated points of \( X \) converging to it. Thus every neighborhood of \( p \) contains some \( x_n \), such that there is a sequence of isolated points of \( X \) converging to \( x_n \). Hence we get a sequence \( \{y_n\}_{n=1}^{\infty} \) of isolated points of \( X \) converging to \( p \). Hence by Lemma 6.2, \( X \) is an almost \( SV \)-space. 

\textbf{Remark.} Recall that a space \( X \) such that whenever \( x \in \text{cl}_X A \) for some \( A \subset X \), there is a sequence of elements of \( A \) that converges to \( x \) is called a \textbf{Frechet space} (or a \textbf{Frechet-Urysohn space}). In the proof of Theorem 6.3, metrizability is used only to produce sequences that converge to points in the closures of some subspaces. It follows that the hypothesis that \( X \) is metrizable can be weakened to assuming only that \( X \) is a Frechet space and \( mC(X) \) is compact. Because \( mC(X) \) fails to be compact if \( X \) is the one-point compactification of an uncountable discrete space, while \( X \) is a compact scattered almost \( SV \)-space if \( \Omega_{SV} \) holds, this new result does not generalize Theorem 6.3.

It follows immediately from Corollary 4 in Section 2.2 of [CD86] that if \( X \) is a metrizable almost \( SV \)-space with a nonisolated point, then \( \Omega_{SV} \) holds. It follows that \( \Omega_{SV} \) holds if and only if there is an infinite compact metrizable almost \( SV \)-space.

6.4 Corollary. If \( [0,1] \) is an almost \( SV \)-space, then \( \Omega_{SV} \) holds.

Ways in which \( C(X) \mod \text{a Prime Ideal Can Be a Valuation Domain} \)

Whether or not the converse of this corollary holds is the most important unsolved problem of this paper. See Section 8.

We conclude this section with a statement without proof of a result that provides some circumstantial evidence that \( (0,1] \) may be an almost \( SV \)-space.

6.5 Theorem. Suppose \( p \in (0,1] \) and \( g \in C([0,1]) \) are such that \( g(p) \neq 0 \) or there is an open set \( U \) of \( (0,1] \) such that \( Z(g) \cap U = \{p\} \). If \( \Omega_{SV} \) holds, then there is a minimal prime ideal \( Q \subset M_p \) such that whenever \( 0 \leq f \leq g \), the coset mod \( Q \) of \( f \) divides the coset mod \( Q \) of \( g \).

Note that it is enough to assume that \( 0 < f \leq g \mod Q \).

This theorem does not enable us to decide whether its conclusion holds in case both \( Z(g) \cap U \) and \( \text{cos}(g) \cap U \) are infinite. If this latter case could be handled, it would follow that \( \Omega_{SV} \) implies \( C([0,1]) \) is an almost \( SV \)-ring.

7. The second generalization; quasi \( SV \)-spaces and rings

The task of determining if a space \( X \) is an almost \( SV \)-space divides naturally into two parts. First we have to find a valuation prime ideal \( P \) contained in a maximal ideal \( M \) of \( C(X) \) and then we have to check whether \( P \) is minimal. This part of the problem is more difficult since there is no easy way to determine whether a prime ideal contained properly in \( P \) is also a valuation prime. This is part of the motivation for the following definition.

7.1 Definition. A space \( X \) such that for each real maximal ideal of \( C(X) \) that is not a minimal prime ideal contains a nonmaximal prime ideal \( P \) such that \( C(X)/P \) is a valuation domain is called a \textbf{quasi \( SV \)-space} (and \( C(X) \) is called a \textbf{quasi \( SV \)-ring}).

In other words \( C(X) \) is a quasi \( SV \)-ring if for all \( p \in \nu X \), whenever \( M_p \neq \text{O}_p \), then \( M_p \) contains a nonmaximal prime ideal \( P \) such that \( C(X)/P \) is a valuation domain.

\textbf{Remark.} The reason for the restriction to real maximal ideals is to make it possible to prove Theorem 7.3 below.

Clearly, every almost \( SV \)-space is a quasi \( SV \)-space. We have been unable to find an example to show that the converse need not hold.

The following lemma will be used in what follows.

7.2 Lemma. If \( P \) is a prime ideal of \( C(X) \) contained in real maximal ideal, then the trace of \( P \) in \( C^*(X) \) is a prime ideal of \( C^*(X) \) and \( C^*(X)/P \cap C^*(X) \cong C(X)/P \).

\textbf{Proof.} Let \( \pi \) be the restriction of \( C(\beta X) \) to \( C(X) \) and \( \pi \) be the canonical homomorphism from \( C(X) \) onto \( C(X)/P \). Now as \( P \) is contained in a real maximal ideal, no element of \( C(X)/P \) is infinitely large and consequently \( \pi \circ \iota \) becomes an epimorphism from \( C(\beta X) \) onto \( C(X)/P \). Thus \( C(\beta X)/P \cong C(X)/P \). Since \( C(\beta X) \cong C^*(X) \), it follows that \( C^*(X)/P \cap C^*(X) \cong C(X)/P \).
7.3 Theorem. For any Tychonoff space X the following are equivalent:
(a) X is a quasi SY-space.
(b) ½X is a quasi SY-space.
(c) ½X is a quasi SY-space.

Proof. (a) and (b) are equivalent since C(X) and C(½X) are isomorphic.
(a) implies (c) Recall that the collection of maximal ideals of C*(X) is given by \( M^* : p \in ½X \); where \( M^* = \{ f \in C^*(X) : f(p) = 0 \} \) and the collection of all maximal ideals of C(X) is given by \( M^p : p \in ½X \); where \( M^p = \{ f \in C(X) : p \in cl_{½X} Z(f) \} \). If \( p \in ½X \) then \( M^p \) becomes a hyperreal maximal ideal of C(X) and hence \( M^p \) properly contains the prime ideal \( M^* \cap C^*(X) \) which is clearly a valuation prime. (See Section 2.1 of [CD86]). Now if \( p \in ½X \) and \( M^* \neq O^* \) then \( M^p \neq O^p \) because if \( M^p = O^p \), then \( M^p = M^p \cap C^*(X) = O^p \cap C^*(X) \), in case \( M^* = O^p \). Since X is a quasi SY-space and \( p \in ½X \), there exists a nonmaximal prime ideal \( P \) contained in \( M^p \) such that \( C(X)/P \) is a valuation domain. Let \( P^* = P \cap C^*(X) \). Then by Lemma 7.2, it follows that \( C^*(X)/P^* \cong C(X)/P \). Because the latter is a valuation domain, \( P^* \) is a nonmaximal valuation prime contained in \( M^p \). Hence \( C^*(X) \) is a quasi SY-ring. Because \( C^*(X) \cong ½X \), the latter becomes a quasi SY-ring and consequently \( ½X \) becomes a quasi SY-space.
(c) implies (a) If \( p \in ½X \) and \( M^p \neq O^p \), there is a nonmaximal prime ideal \( P \) of \( C(X) \) contained in \( M^p \) and by Lemma 7.2, \( C(X)/P \cong C^*(X)/P \cap C^*(X) \). Hence \( P \cap C^*(X) \) is a nonmaximal prime ideal contained in \( M^p \cap C^*(X) \) and evidently \( M^* \neq O^* \). Now since \( ½X \) is a quasi SY-space and \( ½X \cong C^*(X) \), there is a nonmaximal prime ideal \( Q \) containing \( O^* \) in \( C^*(X) \) such that \( Q \) is a valuation prime. We claim that there exists a nonmaximal prime ideal \( W \) of \( C(X) \) containing \( O^* \) such that \( W \cap C^*(X) \) contains \( Q \).

To see this, suppose \( Q_m \subset Q \) is a minimal prime ideal of \( C^*(X) \) containing \( O^* \). As is noted in [HJ65], the mapping that sends each minimal prime ideal of \( C(X) \) to its trace on \( C^*(X) \) is a surjection. So there is a minimal prime ideal \( T_m \) containing \( O^* \) in \( C(X) \) such that \( T_m \cap C^*(X) = Q_m \). Let \( \Omega \) denote the maximal chain of prime ideals containing \( T_m \) in \( C(X) \) and \( \{ T \} \) the collection of all maximal prime ideals of \( C(X) \) which belong to \( \Omega \). Their union \( T \) is a prime ideal of \( C(X) \). Assume that \( T \cap C^*(X) \subset Q \). Now \( M^p \) is a real maximal ideal since \( p \in ½X \). So \( M^p \cap C^*(X) = M^p \), while \( Q \) is a nonmaximal prime ideal of \( C^*(X) \) contained in \( M^p \). This shows that \( T \neq M^p \). Thus \( T \) becomes a prime ideal predecessor of \( M^p \); which implies that \( M^p \) is an upper ideal. But since every maximal ideal is a z-ideal and a z-ideal can never be an upper ideal, this leads to a contradiction. (See Chapter 14 of [GJ76].)

Thus there must exist a (nonmaximal) prime ideal \( W \) of \( \Omega \) such that \( W \cap C^*(X) \) is not contained in \( Q \). Now \( T_m \subset W \) implies \( Q_m \subset W \cap C^*(X) \) and, as we recall \( Q_m \subset Q \). Since the set of prime ideals containing a given prime ideal form a chain, we conclude that \( Q \subset W \cap C^*(X) \). Since being a valuation prime is preserved under extensions, and \( Q \) is a valuation prime of \( C^*(X) \), we conclude that \( W \cap C^*(X) \) is a valuation prime of \( C^*(X) \). Finally, since \( M^p \) is a real maximal ideal of \( C(X) \) and \( W \) is a nonmaximal prime ideal of \( C(X) \) contained in \( M^p \), by Lemma 7.2, \( C(X)/W \cong C^*(X)/W \cap C^*(X) \). Because the latter is a valuation domain, this completes the proof.

Recall that a space \( X \) is recomplete if and only if \( X = ½X \) and that a metrizable space is recomplete if and only if it is of nonmeasurable cardinality. (See Chapters 8 and 12 of [GJ76].)

7.4 Theorem. Every recomplete metrizable space \( X \) is a quasi SY-space if and only if \( \omega \setminus \omega \) holds.

Proof. If \( X \) is a quasi SY-space, there is a \( p \in ½X = X \) such that \( M_p \) contains a nonmaximal prime ideal \( P \) of \( C(X) \) such that \( C(X)/P \) is a valuation domain. Hence by Corollary 4 of Section 2 of [CD86], there is a \( P \)-point in \( ½X \setminus \omega \) and consequently \( \omega \setminus \omega \) holds.

Suppose \( \omega \setminus \omega \) holds, \( p \in X \) is a nonisolated point, and \( \{ x_n \} \) is a sequence of distinct points converging to \( p \) in the metrizable space \( X = ½X \). Clearly \( Y = \{ x_n \} \cup \{ p \} \approx \omega \). Since the restriction mapping \( \varphi \) from \( C(X) \) to \( C(Y) \) is a surjective homomorphism, by \( \omega \setminus \omega \) there exists a non maximal valuation prime ideal \( P \) of \( C(Y) \) and clearly \( C(X)/\varphi^{-1}(P) \cong C(Y)/P \). Because the latter is a valuation domain, \( \varphi^{-1}(P) \) is a nonmaximal valuation prime ideal of \( C(X) \) contained in the real maximal ideal \( M_p \). This completes the proof.

If \( X \) is a quasi SY-space then it certainly follows from Theorem 7.3 that every \( C^* \)-embedded dense subspace of it is again a quasi SY-space. Here is another condition for a subspace of a quasi SY-space to be a quasi SY-space.

7.5 Theorem. Every open recomplete \( C^* \)-embedded subspace \( U \) of a quasi SY-space \( X \) is a quasi SY-space.

Proof. If \( p \in U \) is such that \( M_p^U = \{ f \in C(U) : f(p) = 0 \} \) is not a minimal prime ideal of \( C(U) \), then since \( U \) is open and \( C^* \)-embedded in \( X \), we will show that \( M_p = \{ f \in C(X) : f(p) = 0 \} \) is not a minimal prime ideal of \( C(X) \).

For, by assumption, there is a \( f \in M_p^U \), that does not vanish on a neighborhood of \( p \) in \( U \). Because \( U \) is \( C^* \)-embedded and open, \( f \) has a continuous extension that is in \( M_p \setminus O_p \). So \( M_p \) is not in \( mC(X) \).

Since \( X \) is a quasi SY-space, \( M_p \) contains a nonmaximal valuation prime ideal \( P \) of \( C(X) \). If \( i \) denotes the restriction map of \( C(X) \) onto \( C(U) \), then since \( U \) is open, \( ker(i) \subset O_p \subset P \). Thus \( P/ker(i) \) becomes a prime ideal of \( C(X)/ker(i) \) and clearly \( C(X)/P \cong C(U)/i(P) \). Since \( i \) is an epimorphism, \( C(X)/ker(i) \cong C(U) \) and \( P/ker(i) \cong i(P) \), and moreover, \( i(P) \) is a prime ideal of \( C(U) \). Thus \( C(X)/P \cong C(U)/i(P) \). As the former is a valuation domain, \( i(P) \) becomes a nonmaximal valuation prime ideal of \( C(U) \) contained in \( M_p^U \), and because \( U \) is recomplete, this completes the proof.
7.6 Theorem. If a realcompact space $X$ can be expressed as an arbitrary union of open $C$-embedded subspaces such that each of them is a quasi $SV$-space, then $X$ is a quasi $SV$-space.

Proof. Suppose $X$ is realcompact and $\{X_\alpha\}$ is a collection of open $C$-embedded quasi $SV$-subspaces such that $X = \bigcup X_\alpha$. If $p \in X$, then $p \in X_\alpha$ for some $\alpha$. As in the proof of 7.5, let $M_p^{X_\alpha} = \{f \in C(X_\alpha) : f(p) = 0\}$. We will show that if $M_p$ is not a minimal prime ideal of $C(X)$, then the same assertion will hold for $M_p^{X_\alpha}$ in $C(X_\alpha)$.

For if there is an $f \in M_p \setminus O_p$ and $M_p^{X_\alpha} = O_p^{X_\alpha}$, then $f|X_\alpha$ vanishes on a neighborhood of $p$ in the open subset $X_\alpha$ in $X$. Thus $M_p$ is also minimal prime in $C(X)$. If $i$ is the restriction of $C(X)$ onto $C(X_\alpha)$, then since $X_\alpha$ is a quasi $SV$-space, there exists a nonmaximal valuation prime ideal $P$ of $C(X_\alpha)$ contained in $M_p^{X_\alpha}$ and $C(X)/i^{-1}(P) \cong C(X_\alpha)/P$. Thus $i^{-1}(P)$ is a nonmaximal valuation prime ideal of $C(X)$ contained in $M_p$. This completes the proof. □

7.7 Theorem. Finite products of compact quasi $SV$-spaces are quasi $SV$-spaces.

Proof. It suffices to prove that the product $X \times Y$ of two quasi $SV$-spaces is a quasi $SV$-space. Each maximal ideal of $C(X \times Y)$ is in the set $\{M_{(p,q)} : (p,q) \in X \times Y\}$. If $(p,q) \in X \times Y$ is a nonisolated point, then either $p$ or $q$ is a nonisolated point of $X$ and $Y$ respectively. Assume $p$ is a nonisolated point of $X$. If $W$ denotes the space $X \times \{q\}$, then $W \approx X$. If $i$ is the restriction mapping from $C(X \times Y)$ onto $C(W)$, then since $X$ is quasi $SV$-space, there exists a nonmaximal valuation prime ideal $P$ of $C(W)$ contained in the maximal ideal $\{f \in C(W) : f(p,q) = 0\}$ of $C(W)$. Since $i$ is an epimorphism, $(C(X \times Y))/i^{-1}(P) \cong C(W)/P$. Because the latter is a valuation domain, $i^{-1}(P)$ is a nonmaximal valuation prime ideal of $C(X \times Y)$ contained in $M_{(p,q)}$. Since $(p,q)$ is an arbitrary nonisolated point, this completes the proof. □

7.8 Definition. A point $p \in \beta X$ is a $Qsv$-point if $M_p$ contains a nonmaximal valuation prime of $C(X)$ that is a $z$-ideal.

7.9 Examples

1. If every $p \in \beta X$ such that $M_p$ is a real maximal ideal of $C(X)$ is a $Qsv$-point, then $X$ is a quasi $SV$-space.
2. Every point of a compact $F$-space that is not a $P$-point is a $Qsv$-point.
3. If $\Omega sv$ holds, then every nonisolated point $p$ of a metrizable space $X$ is a $Qsv$-point.

For, since $\Omega sv$ holds, there is a minimal valuation prime $P$ of $C(\omega)$ contained in $M_p$. If $i$ is the restriction of $C(X)$ onto $C(\omega)$, then $i^{-1}(P)$ is a nonmaximal valuation prime $z$-ideal of $C(X)$ contained in $M_p$ and consequently $p$ is a $Qsv$-point.

Our next result is a sufficient condition for a compact, perfectly normal space to be a quasi $SV$-space.

7.10 Theorem. Suppose $X$ is compact and perfectly normal. If, for every nonisolated point $p$ of $X$, there exists a free $P(p)$-ultrafilter $\mathcal{Z}[M_p]$ on $X \setminus \{p\}$ such that $q$ is a $Qsv$-point of $\beta(X \setminus \{p\})$, then $X$ is a quasi $SV$-space.

Proof. Their hypothesis implies that every $p \in X$ is a $G_\delta$-point, every maximal ideal of $C(X)$ is real and that there is a free maximal ideal $M_c$ of $C(Y)$ which contains a nonmaximal prime $z$-ideal $Q$ of $C(Y)$ which is a valuation prime, where $Y = X \setminus \{p\}$. By 4.3(b)(2), $\gamma(M_c)$ becomes an immediate prime $z$-ideal predecessor of the maximal ideal $M_p$ of $C(X)$ in the class of all all $z$-ideals. (Recall that $\gamma$ satisfies $Z(\gamma(Q)) = \{\text{on} : Y \in \mathcal{Z}[M_c]\} = \{\text{on} : Y \in \mathcal{Z}[M_c]\}$.) By Corollary 2.3.3 of [CD86], $\gamma(M_c)$ becomes a valuation prime. Because the prime $z$-ideal $Q$ of $C(Y)$ is contained in $M_c$, it is clear that $Z(\gamma)$ converges to $q$. Therefore by 4.3(b)(1), $\gamma(Q)$ becomes a prime $z$-ideal of $C(X)$ contained in the maximal ideal $M_p$. Clearly $\gamma(Q) \subseteq \gamma(M_c)$. By Theorem 2.2.2 in [CD86], it follows that if $P$ is properly contained in $M_p$ and is a valuation prime, then $Z(T)$ is a $(p)$-filter for every maximal prime ideal $T$ contained in $P$. Since any $z$-filter containing a $P(p)$-filter is again a $P(p)$-filter, it follows that $Z(\gamma(Q))$ is a $(p)$-filter. By the definition of the mapping ‘$\gamma$’, it follows that $Z(\gamma(Q)) = (p)$-filter if and only if $Z(Q)$ is. Finally, since $Q$ is a valuation prime of $C(Y)$ and $Z(Q)$ is a $(p)$-filter, it follows from Theorem 4.9 that $\gamma(Q)$ is a valuation prime contained in $M_p$. Since $p$ is an arbitrary nonisolated point of $X$, this completes the proof. □

We conclude this section with two results concerning chains of pseudoprime ideals.

$P$ is called a primary ideal of a commutative ring if $ab \in P$ implies either $a$ or some power of $b$ belongs to $P$ and is called pseudoprime if $ab = 0$ implies either $a$ or $b$ belongs to $P$. It is well known that every prime ideal is primary and every primary ideal is pseudoprime. While it need not hold for arbitrary commutative rings, in a ring $C(X)$, an ideal of $C(X)$ is pseudoprime if and only if it contains a prime ideal. (See [GK60] and [G90].)

If $I$ is an ideal of a commutative ring $A$ and $f \in A$, then $(I,f)$ denotes the smallest ideal of $A$ containing $I$ and $f$, while $(I,f)$ denotes the principal ideal of $A/I$ generated by the coset $f + I$.

7.11 Theorem. If $X$ is a topological space and $p \in X$ is a $Qsv$-point then there exists a countable chain of pseudoprime ideals of $C(X)$ contained in $M_p$ which are not primary ideals.

Proof. As $p \in X$ is a $Qsv$-point, there is a prime ideal $P$ of $C(X)$ properly contained in $M_p$ which is a valuation prime ideal. If $f_1 \in M_p \setminus P$, then $(P,f_1)$ is a pseudoprime ideal since it contains the prime ideal $P$. Because, as is shown in [K58], no proper principal ideal of $C(X)/P$ is primary, the principal ideal $(P,f_1)$ is not a primary ideal of $C(X)/P$ and consequently $(P,f_1)$ is not a prime ideal of $C(X)$ because $(P,f_1)/P = (P,f_1)$. Now $(P,f_1) \neq M_p$ since $(P,f_1)$ is not primary. So there is an $f_2 \in M_p \setminus (P,f_1)$. Because $C(X)/P$ is a valuation domain, one of $(P,f_1)$ or $(P,f_2)$ must divide
the other. If \( P(f_1) \mid P(f_2) \) then \( f_2 \in (P, f_1) \) — which is a contradiction. Therefore \( P(f_2) \mid P(f_1) \). Hence \( (P(f_1)) \subseteq (P(f_2)) \) and consequently \( (P, f_1) \subseteq (P, f_2) \). Then there is an \( f_3 \in M_f \setminus (P, f_2) \). Continuing this process, we get a countably infinite chain of pseudoprime ideals \( (P, f_1) \subseteq (P, f_2) \subseteq \cdots \subseteq (P, f_n) \subseteq \cdots \) which are not primary.

Recall from Definition 4.7 that a point \( p \) of \( \beta X \) such that \( O^p \) is a valuation prime is called a special \( \beta F \)-point. An algebraic characterization of such points follows.

**7.12 Theorem.** A point \( p \in \beta X \) is a special \( \beta F \)-point if and only if the pseudoprime ideals of \( C(X) \) containing \( O^p \) that are not primary form a chain (i.e., are linearly ordered under set inclusion).

**Proof.** If \( p \) is a special \( \beta F \)-point then \( O^p \) is valuation prime and therefore the principal ideals of \( C(X)/O^p \) form a chain. This is equivalent to the fact that the ideals of \( C(X)/O^p \) form a chain and consequently the ideals of \( C(X) \) containing \( O^p \) form a chain. In particular, the pseudoprime ideals containing \( O^p \) that are not primary form a chain.

Suppose \( p \) is not a \( \beta F \)-point. Then consider two distinct maximal chains \( \Phi \) and \( \Psi \) of prime ideals lying between \( O^p \) and \( M^p \) in \( C(X) \). Suppose \( P \) and \( Q \) are the minimal prime ideals in \( \Phi \) and \( \Psi \) respectively. Now \( \Phi \cap \Psi = P + Q \) is the minimal member of the (intersecting) chain \( \Phi \cap \Psi \), which is a prime \( q \)-ideal. Since \( P \cup Q \) is not an ideal, there is an \( f \in P + Q \setminus (P \cup Q) \).

Now \( (P, f) \) and \( (Q, f) \) are pseudoprime ideals since they contain prime ideals. Neither of them is primary as by the proof of 7.11. Suppose it were the case that \( (P, f) \subseteq (Q, f) \). Then \( P + Q = (Q, f) \) since \( (Q, f) \subseteq P + Q \). Because \( (Q, f) \) is not a primary ideal while \( P + Q \) is prime, we arrive at a contradiction. Hence neither of \( (P, f) \) or \( (Q, f) \) is contained in the other.

8. Remarks and problems

In this section, we refer readers to some papers concerned with residue class rings of the form \( C(X)/P \) for \( P \) a prime ideal of \( C(X) \) whose content we have been able to use only to a very limited extent. These papers inspired us to pose some interesting problems and to derive a few results. Our hope is that some of our readers may be able to make better use of them. In the long and thorough paper [M90], James Moloney examined closely the residue class domains of \( C(\omega) \), \( C(\infty)(R) \), and to a lesser extent \( C(X) \) (for some other classes of topological spaces) modulo prime ideals assuming CH.

His extraordinary and difficult accomplishment is showing that

\[
\{C(\omega)/P : P \text{ a nonmaximal prime ideal}\}
\]

was divided into precisely 9 distinct isomorphism classes. (Any two members of the same class are isomorphic, and no two distinct classes contain members that are isomorphic.) The descriptions of these isomorphism classes are order theoretic without any direct description of the algebraic properties of these integral domain. Instead, they involve the cardinality and nature of cofinal and coinitial subsets. There does seem to be a way of describing these results succinctly. The curious reader should examine Theorem 3.2.26 of [M90] and each of the theorems referred to in its proof. No attempt is made to determine when \( C(\omega)/P \) is a valuation domain. We do know, however, from Theorem 3.5(d) that not every prime ideal contained in \( M_\omega \) is a valuation prime, and that (assuming CH) \( M_\omega \) contains a valuation prime because CH implies \( \aleph_2 \). In Section 4 of [M90], some of the results referred to above are applied to some more general spaces. Regrettably, Moloney's interesting results are of little help to us because our goals are different from his. For example, we do not know how to tell which of the equivalence classes described above contains an element \( C(\omega)/P \) such that \( P \) is a valuation prime and \( P \subseteq M_\omega \). If we could answer the following questions, we might be able to use some of Moloney's results to reach our main goals. In each case we assume CH.

8.1 Problems

(a) Which of Moloney's 9 equivalence classes contains an element \( C(\omega)/P \) such that \( P \) is a valuation prime?

(b) If \( P \) is a nonmaximal valuation prime of \( C(\omega) \), can the set of strictly positive elements of \( C(\omega)/P \) have a countable cofinal subset?

Assuming CH, two problems less related to [M90] are:

8.2 Problems

(a) Is \([0, 1]\) an almost \( SV \)-space?

(b) Is every compact metrizable scattered space with finite CB-index an almost \( SV \)-space?

Note that by Theorem 7.4 (and CH), the spaces above are quasi \( SV \)-spaces.

In [J60], the authors pose the question:

(*) If \( Q \) is a prime ideal of \( C(Y) \), when is there a space \( X \) and a maximal ideal \( M \) of \( C(X) \) such that \( C(X)/M \) and the quotient field of \( C(Y)/Q \) are isomorphic?

When (*) has an affirmative answer, they say that \( C(Y)/Q \) is realized by \( C(X)/M \). It is shown in Theorem 2.3 of [J60] that if \( C(Y)/Q \) has a realization, then \( Q \) is a \( z \)-ideal.

8.3 Definition. A Tychonoff space \( Y \) and the ring \( C(Y) \) is said to be prime \( z \)-space if each nonmaximal prime \( z \)-ideal has an immediate successor in the set of prime \( z \)-ideals.

Note that any space in which any chain of prime \( z \)-ideals is finite is prime \( z \)-space. Spaces with this finiteness property are studied in [HMW03] and [M05]. In particular, the one-point compactification of an infinite discrete space is prime \( z \)-space. In Example 4.3 of [J60], it is shown that \([0, 1]^\omega\) is not prime \( z \)-space. By the \( z \)-dimension of a space \( X \), we mean the supremum of the lengths of chains of prime \( z \)-ideals of \( C(X) \). In Section 5 of [M05], it is shown that if a compact space \( X \) is scattered, then \( C(X) \) has finite \( z \)-dimension if and only its CB index
is finite. (For a precise definition of z-dimension and its properties with emphasis on the case when X is compact, see Sections 4 and 5 of [M208].)

8.4 Theorem. If Y is a compact space that is prime z-sparse, and Q is a minimal nonmaximal prime ideal of C(Y) such that C(Y)/Q is a valuation domain, then its set of strictly positive elements has no countable coinitial subset.

Proof. Suppose Q is as above. By Theorem 3.4 of [GJ60], there is a subspace X of Y and a maximal ideal M of C(X) such that C(X)/M and the quotient field of C(Y)/Q are isomorphic. Because Q is not maximal, the field C(Y)/Q is an \eta_1-set. (See Chapter 13 of [GJ76].) Hence its set of strictly positive elements has no countable coinitial subset. It follows easily that the valuation domain C(Y)/Q has the property as well.

The proof of the following corollary follows from the last theorem and the remarks preceding it.

8.5 Corollary. If Y is a compact scattered space with finite CB-index, and Q is a minimal valuation prime ideal of C(Y) such that C(Y)/Q is not a field, then its set of strictly positive elements of C(Y)/Q has no countable coinitial subset.

Next, we include with some brief remarks about the contents of [Sc97].

Note first that the term real closed ring is used by Schwartz in an entirely different way than in [CD96]. Because Schwartz's terminology is used in many papers, we will use it in what follows. We will not repeat the definition of real closed ring. It will be enough for the reader to know that a real closed ring is a lattice ordered ring, that any C(X) is real closed, and that C(X)/P is real closed ring whenever P is a prime ideal of C(X). If M is a maximal ideal of C(X), let P(M) denote set of prime ideal of C(X) that are contained in M. The author explores the relationship between C(X) being an SV-ring and \{(X)/Q : Q \in P(M)\} consisting of valuation domains for a collection of maximal ideals M of C(X). We have been unable to adjust this approach to the study of almost SV-spaces, but hope that some readers of this paper may be able to do so.

In Chapter 4 of the Dales-Woodin book [DW96], these authors study the residue class rings C(X)/P with which we are concerned in case X is compact. This chapter is not self-contained and the notation used in it differs not only from what we use, but also from many of the articles to which the reader is referred.

If P is a prime ideal of C(X), where X is compact and M are the unique maximal ideal of C(X) containing P, then P is called strongly convex if \(M_P/P\) is an interval in the quotient field of C(X)/P. The notion of a strongly convex prime ideal may play a major role in studying the valuation prime ideals of C(X) because every valuation prime ideal is necessarily a strongly convex prime. Just studying strongly convex primes will not suffice since the converse of this latter assertion need not hold. For, it is shown in the proof of Theorem 4.40 of [DW96] that there is a nonmaximal valuation prime ideal P of C(βω × αω) while there exists a prime ideal Q \(\subseteq P\) which is not even strongly convex and hence fails to be valuation prime.

Actually, there is a much simpler example. By Prop. 4.36 of [DW96], every prime ideal of C(αω) is strongly convex. But not every prime ideal of C(αω) is a valuation prime since αω is not an SV-space. In Proposition 4.37 of [DW96], it is shown that if P is a strongly convex prime ideal of C(X) and X is compact, then the quotient field of C(X)/P is a semi-\eta_1 field. That is, whenever every element of A is less than every element of B, where A is an increasing and B is a decreasing (countable) sequence of elements of the quotient field of C(X)/P, there is a \(a \in A\) in the quotient field of C(X)/P strictly between A and B. Note that R is a semi-\eta_1 field that is not an \eta_1-set. It follows from Prop 2.20 of [DW96] assuming CH that if in a semi-\eta_1-field the minimum cardinality of a cofinal (or coinitial) subsets of strictly positive elements is \(\geq \aleph_1\), then it is not an \eta_1-field.

The discussion above leads us to believe that a more careful study of the properties of strongly convex ideals and related topics in [DW96] may lead to solutions of some of the problems posed above. We hope this is the case despite the fact that the main focus of this book is on the nature of quotient fields of rings C(X)/P where P is a nonmaximal prime ideal of C(X). The latter are the super-real fields of the title.

Added in proof. We can improve Theorem 6.3 by showing that

Every compact metrizable scattered space is an almost SV-space if \(\Omega_{\alpha}\) holds.

References


Ways in which $C(X)$ mod a Prime Ideal Can be a Valuation Domain


[L03] S. Larson, Constructing rings of continuous functions in which there are many maximal ideals of nontrivial rank, Comm. Alg 31 (2003), 2233–2260.


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