Solution Set 2

1. **Review of Div, Grad and Curl.** Prove:

1. (a) \( \nabla \cdot (\nabla \times A) = 0 \), where \( A \) is any three dimensional vector field.

\[
\nabla \times A = \begin{vmatrix}
 i & j & k \\
 \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
 A_x & A_y & A_z
\end{vmatrix} = i \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + j \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + k \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)
\]

\[
\nabla \cdot (\nabla \times A) = \frac{\partial}{\partial x} \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \frac{\partial}{\partial y} \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)
\]

= 0 by equality of mixed partials.

2. (b) \( \nabla \times \nabla u = 0 \), where \( u \) is any three dimensional scalar field (i.e. \( u = u(x, y, z) \)).

\[
\nabla \times \nabla u = \begin{vmatrix}
 i & j & k \\
 \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\
 \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^2 u}{\partial z^2}
\end{vmatrix} = i \left( \frac{\partial^2 u}{\partial z \partial y} - \frac{\partial^2 u}{\partial y \partial z} \right) + j \left( \frac{\partial^2 u}{\partial z \partial x} - \frac{\partial^2 u}{\partial x \partial z} \right) + k \left( \frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial x \partial y} \right)
\]

= 0

2. **Orthogonality.** Do problem 2.3.5. in Haberman. Note that the object of this problem is to prove that sines are orthogonal. Therefore you must actually evaluate the integrals rather than invoking orthogonality. **Hint:** For the case \( n = m \) use the identity \( \sin^2 \theta + \cos^2 \theta = 1 \) and notice that \( \int_0^\pi \sin^2 \theta \, d\theta = \int_0^\pi \cos^2 \theta \, d\theta \).

\[
\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = \int_0^L \left[ \frac{\sin n\pi x}{L} \right] \left[ \frac{\sin m\pi x}{L} \right] \, dx
\]

\[
= \frac{1}{2} \left[ \frac{\pi}{n (n-m)} \sin \frac{n\pi x}{L} \left( \frac{\sin n\pi x}{L} - \frac{\sin (n\pi x + m\pi x)}{L} \right) \right]_0^L
\]

= 0 for \( m \neq n \)

For \( m = n \):

\[
\int_0^L \sin^2 \frac{m\pi x}{L} \, dx = \frac{1}{2} \left[ \int_0^L \sin^2 \frac{m\pi x}{L} \, dx + \int_0^L \cos^2 \frac{m\pi x}{L} \, dx \right]
\]

\[
= \frac{1}{2} L \int_0^L 1 \, dx = \frac{L}{2}
\]

3. **Heat Equation on a Thin Circular Wire.** Do problem 2.4.6. in Haberman.

(a) At equilibrium, \( \frac{\partial^2 u}{\partial x^2} = 0 \). Solve ODE with a polynomial: \( u = c_1 x + c_2 \). Periodic boundary conditions \( \Rightarrow c_1 = 0 \). Since there is no flux out of the system the final energy must be the same as the initial energy thus \( c_2 \) is the average of the initial conditions.

\[
u(x) = c_2 = \frac{1}{2L} \int_{-L}^L f(x) \, dx.
\]
(b) 

\[ u(x, t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} e^{-(n\pi/L)^2 kt} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} e^{-(n\pi/L)^2 kt} \]

as \( t \to \infty \), \( u(x, t) \to a_0 \). Therefore

\[ u(x, t) \approx a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx. \]

4. **Separation of Variables, Laplace’s Equation.** Solve problem 2.5.3. in Haberman. See Haberman.

5. **“Minimum Principles.”** Solve problem 2.5.13. in Haberman.

See page 80 in Haberman. Suppose the minimum occurs at point \( P \). By the Mean Value Theorem, the value at \( P \) must be the average of the points on a circle around \( P \). This is impossible if \( u(P) \) is less than the value of \( u \) at all the points on the circle. Thus, by contradiction, the minimum must be on the boundary.

6. **Wandering Bacteria.** Derive a partial differential equation to describe the evolution of the concentration of bacteria assuming that at every time step, each bacteria has a equal probability of moving to the left or moving to the right or staying put. (Recall that in class we did this problem assuming equal probability of moving left or right and zero probability of not moving.)

\[ c(x, t + \Delta t) = \frac{1}{3} c(x + \Delta x, t) + \frac{1}{3} c(x, t) + \frac{1}{3} c(x - \Delta x, t) \]

Taylor Series:

\[ c + \Delta t \frac{\partial c}{\partial t} + \ldots = \frac{1}{3} \left( c + \Delta x \frac{\partial c}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 c}{\partial x^2} + \ldots \right) \frac{1}{3} c + \frac{1}{3} \left( c - \Delta x \frac{\partial c}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 c}{\partial x^2} + \ldots \right) \]

\[ \frac{\partial c}{\partial t} = \frac{2}{3} \frac{\Delta x^2}{\Delta t} \frac{\partial^2 c}{\partial x^2} \Rightarrow \frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} \]

Same as before except diffusion coefficient is smaller (i.e. it takes longer for the bacteria to diffuse).

7. **Conservation of Mass.** Do problems 2.5.17. and 2.5.18. in Haberman.

1D:

\[ \frac{\partial}{\partial t} (\text{mass in slice}) = \text{mass flowing in} - \text{mass flowing out} \]

\[ \frac{\partial}{\partial t} (\rho A \Delta x) = \rho(x) u(x) A - \rho(x + \Delta x) u(x + \Delta x) A \]

\[ \frac{\partial \rho}{\partial t} = \frac{\rho(x) u(x) - \rho(x + \Delta x) u(x + \Delta x)}{\Delta x} = -\frac{\partial}{\partial x} (\rho u) \]
3D:
\[
\frac{\partial}{\partial t} \int \int \int \rho \, dV = - \oint \rho \mathbf{u} \cdot \mathbf{n} \, dS = - \int \int \int \nabla \cdot (\rho \mathbf{u}) \, dV
\]

This is only true if the integrand is zero.

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0
\]

If \( \rho \) is constant: \( \nabla \cdot (\rho \mathbf{u}) = 0 \Rightarrow \nabla \cdot \mathbf{u} = 0. \)

8. **Flow Past a Cylinder.** Do problem 2.5.24. in Haberman.

\[
\mathbf{u}_\theta = -c_1 \frac{a}{r} - U \left(1 + \frac{a^2}{r^2}\right) \sin \theta
\]

At the surface of the cylinder, \( \mathbf{u}_\theta = -\frac{c_1}{a} - 2U \sin \theta \). When the circulation is negative, \( c_1 \) is positive. At the top of the cylinder (\( \theta = \frac{\pi}{2} \)) \( \mathbf{u}_\theta = -\frac{c_1}{a} - 2U \) and at the bottom \( \mathbf{u}_\theta = -\frac{c_1}{a} + 2U \). Thus the fluid is moving fastest (and backwards) at the top.

---

**Matlab 2. Fourier series. How good is it?**

For the following functions, plot the function and the first \( M \) terms of the Fourier series where \( M = 1, 2, 4, 8 \). (Make one plot for each function listed below with all five curves so you can see how well each approximation compares with the original).

(a) \( f(x) = \sin^3 x \) on the interval \( 0 \leq x \leq 2\pi \)

(b) \( f(x) = \begin{cases} 1 & x \leq 1 \\ 0 & x > 1 \end{cases} \) on the interval \( 0 \leq x \leq 2 \)

(c) \( f(x) = |x| \) on the interval \( -1 \leq x \leq 1 \)

Comment on which functions seem to be approximated best by the Fourier series. Which series appears to converge to the actual function fastest? Why?

```matlab
%% Fourier Series

%% y = step function

figure(2)
N = 40;
x = linspace(0, 2, N);
y(1:N/2) = 1;
y(N/2:N) = 0;
plot(x,y)
hold on
FS = ones(1,40)/2.;
```
for i = 1:8  
a(i) = 0.0;  
b(i) = (cos(i*pi) + 1 - 2.*cos(i*pi/2.))/(i*pi);  
FS = FS + cos(i*pi*x/2.)*a(i) + sin(i*pi*x/2.)*b(i);  
plot(x,FS,'ro')  
end  
axis([0 2 -0.5 1.5])  
xlabel('x')  
ylabel('y = step function')  
title('Fourier Series (First 8 Terms)')  
hold off  

Similarly for others except $a$'s and $b$'s become:

```matlab
%% y = sin^3(x) %%

a(i) = 0.0;  
b = [0 3./4. 0 0 0 -1./4. 0 0];

%% y = |x| %%

a(i) = 2./(i^2*pi^2)*(cos(i*pi)-1);  
b(i) = 0.0;
```
Note that the smoother the function is, the better the Fourier approximation.