1.1.14 Prove that removing opposite corner squares from an 8-by-8 checkerboard leaves a subboard that cannot be partitioned into 1-by-2 and 2-by-1 rectangles. Using the same argument, make a general statement about all bipartite graphs.

We define a graph $G$ with

- a vertex representing each square of the board,
- and two vertices are adjacent if and only if the corresponding squares share an edge.

Since squares of checkerboard can be alternately colored black and white across rows and down columns, the coloring of squares partitions the vertices of $G$ into two independent sets. Thus, $G$ is bipartite. Since the two removed squares are at opposite corners, they would have had the same color, so the parts in the bipartition of $G$ have sizes 30 and 32.

Covering the board with dominoes is the same as finding a set of pairwise disjoint edges in $G$ that match up all vertices of $G$ (i.e., the edges of a 1-regular spanning subgraph of $G$.) But since $32 > 30$, it is impossible to find such a collection of pairwise disjoint edges, as each edge “covers” exactly two vertices, one from each part of the bipartition.

A generalization is that the vertices in the two parts of a bipartition of any bipartite graph cannot be matched up using pairwise disjoint edges if the two parts have unequal sizes.

1.1.15 Consider the following four families of graphs: $A = \{ \text{paths} \}$, $B = \{ \text{cycles} \}$, $C = \{ \text{complete graphs} \}$, $D = \{ \text{bipartite graphs} \}$. For each pair of these families, determine all isomorphism classes of graphs (i.e., all unlabeled graphs) that belong to both families.
We have the following intersections:

\[ A \cap B = \emptyset \] since a cycle has an equal # of edges and vtc., while a path does not.

\[ A \cap C = \{ K_1, K_2 \} \] since paths contain no cycles, and \( K_n \) for \( n \geq 3 \) has cycles.

\[ A \cap D = \{ \text{paths} \} \] since every path is a bipartite graph.

\[ B \cap C = \{ C_3 \} \] since \( |E(K_n)| = \binom{n}{2} \) and \( |E(C_n)| = n \), the only possibility is \( n = 3 \), and indeed \( K_3 \cong C_3 \).

\[ B \cap D = \{ C_n : n \geq 4, n \text{ is even} \} \] since odd cycles are not bipartite.

\[ C \cap D = \{ K_1, K_2 \} \] since bipartite graphs cannot contain \( K_3 \).

1.1.23(c) Among simple graphs, determine the smallest \( n \) such that there exist nonisomorphic \( n \)-vertex graphs having the same list of vertex degrees.

The smallest such \( n \) is \( n = 5 \). Two nonisomorphic graphs on 5 vertices, both with degree sequence \( 1, 1, 2, 2, 2 \), are given in Figure 1. It is immediately clear that the two graphs are not isomorphic since one is connected and one is disconnected.

![Figure 1](image)

Figure 1: Two nonisomorphic graphs on \( n = 5 \) vertices.

All 11 nonisomorphic graphs on four vertices are given on page 11 of the text, so we see that there are no two graphs with the same degree sequence among this collection. (Similar analysis can be done for graphs with one, two, and three vertices, or one can argue that once we know \( n \neq 4 \), it follows that \( n \not< 4 \) using an argument with isolated vertices.)

1.1.31 A graph \( G \) is self-complementary if it is isomorphic to its complement. Examples of self-complementary graphs include \( P_4 \) and \( C_5 \).

Prove that a self-complementary graph with \( n \) vertices exists if and only if \( n \) or \( n - 1 \) is divisible by 4. (Hint: When \( n \) is divisible by 4, generalize the structure of \( P_4 \) by splitting the vertices into four groups. For \( n \equiv 1 \mod 4 \), add one vertex to the graph constructed for \( n - 1 \).)
Suppose a self-complementary graph $G$ on $n$ vertices exists. Then $G$ and $\overline{G}$ must have the same number of edges since they are isomorphic. We also know that together they have $\binom{n}{2}$ edges with no edge repeated in both $G$ and $\overline{G}$. So it must be that

$$|E(G)| = |E(\overline{G})| = \frac{1}{2} \binom{n}{2} = \frac{n(n-1)}{4}.$$ 

Clearly the number of edges must be an integer, so $n(n-1)$ is divisible by 4. Since $n$ and $n-1$ are not both even (i.e., it is not the case that $2|(n-1)$ and $2|n$), it follows that $n$ or $n-1$ must be divisible by 4.

$(\Leftarrow)$ Suppose $n = 4k$ for some positive integer $k$. We construct an $n$-vertex self-complementary graph. Define four sets of vertices $V_1, V_2, V_3, V_4$, each of size $k$, and let $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$. Let $E(G)$ contain the following edges:

$$\{uv : u \in V_1, v \in V_2\} \cup \{uv : u \in V_2, v \in V_3\} \cup \{uv : u \in V_3, v \in V_4\}.$$

To specify the remaining edges, induce copies of any graph $H$ on the $k$ vertices of $V_1$ and those of $V_4$. (For example, $H$ may be the complete graph $K_k$.) Within the vertices of $V_2$ and those of $V_3$, induce $\overline{H}$. This construction yields a graph $G$ such that $G$ and $\overline{G}$ are isomorphic, which can be seen in Figure 2. (Bold black lines indicate all possible edges exist between two subsets of vertices.)

![Figure 2: General structure of a self-complementary graph on $n = 4k$ vertices.](image)

For $n = 4k + 1$, form graph $G$ by adding a vertex $v$ to the graph previously constructed. Join $v$ to all $k$ vertices of $V_1$ and all $k$ vertices of $V_4$. The isomorphism showing that $G - v$ is self-complementary also shows that $G$ is self-complementary by mapping vertex $v$ to itself. See Figure 3.

![Figure 3: General structure of a self-complementary graph on $n = 4k + 1$ vertices.](image)

1.2.22 Prove that a graph $G$ is connected if and only if for every partition of its vertices into two nonempty sets, there is an edge with endpoints in both sets.
(⇒) Suppose $G$ is connected. Given a partition of $V(G)$ into sets, $S$ and $T$, let $u \in S$ and $v \in T$. Then there exists a $u, v$-path $P$ since $G$ is connected. After the last vertex of $P$ in $S$, there is an edge from $S$ to $T$.

(⇐) We prove the contrapositive—suppose $G$ is disconnected. Let $H$ be a component of $G$. Consider the partition of the vertex set into $V(H)$ and $V(G) - V(H)$. (Since $G$ is disconnected, these are both nonempty sets of vertices.) There is no path from any vertex in $H$ to any vertex in $G - H$, so there is no edge with endpoints in both $V(H)$ and $V(G) - V(H)$.

Alternate proofs:

(⇐, proof by extremality) Consider an arbitrary vertex $v \in V(G)$, and let $S$ be the set of all vertices reachable from $v$ via paths. We claim that $S = V(G)$—suppose not. Then consider the partition $S$ and $V(G) - S$ of the vertex set. By assumption, there exists an edge with an endpoint $x \in S$ and an endpoint $y \in V(G) - S$. Now there exists a $v, y$-path by extending a $(v, x)$-path along edge $xy$. $\Rightarrow \Leftarrow$ So $S = V(G)$, and it follows that $G$ is connected, since any two vertices both connected to $v$ must then be connected to each other (by the transitivity of is-connected-to relation).

(⇐, algorithmic) We “grow” a set of vertices that are in the same equivalence class of the is-connected-to relation. Start with one vertex in $S$. By assumption, there exists an edge with endpoints $x \in S$ and $y \notin S$. Adding $y$ to $S$, we have increased the size of $S$ and all vertices in $S$ are connected to each other (by transitivity of the is-connected-to relation). Repeat this procedure until no vertices lie outside of $S$, implying that $G$ has only one component.

1.3.17 Let $G$ be a graph with at least two vertices. Prove or disprove:

(a) Deleting a vertex of degree $\Delta(G)$ cannot increase the average degree.

(b) Deleting a vertex of degree $\delta(G)$ cannot reduce the average degree.

(a) The statement is TRUE. Let $G$ be a graph with $n \geq 2$ vertices and $m$ edges. Let $v$ be a vertex in $G$ of maximum degree, $\Delta(G)$. Then, using the degree-sum formula, we have

$$\text{avg degree in } G = \frac{1}{n} \sum_{u \in V(G)} d_G(u) = \frac{2m}{n}$$
and

\[
\text{avg degree in } G - v = \frac{1}{n-1} \left( \sum_{u \in V(G) - v} d_{G-v}(u) \right)
\]
\[
= \frac{1}{n-1} \left( \sum_{u \in V(G)} d_G(u) - 2\Delta(G) \right)
\]
\[
= \frac{2m - 2\Delta(G)}{n-1}
\]
\[
\leq \frac{2m - 2m}{n-1} \frac{n}{n-1} \quad \text{since } \Delta(G) \geq \text{avg degree of } G = \frac{2m}{n}
\]
\[
= \left( \frac{2m}{n} \right) \left( \frac{n-2}{n-1} \right)
\]
\[
< \frac{2m}{n}.
\]

(b) The statement is FALSE. Consider the path on 3 vertices, \( P_3 \). The average degree is \( \frac{1}{3}(1 + 1 + 2) = \frac{4}{3} \) but for a leaf \( v \in V(P_3) \), the average degree of \( P_3 - v = P_2 \) is \( \frac{1}{2}(1 + 1) = 1 \). So the average degree can, in fact, be reduced when a vertex of minimum degree is removed.

Note that any \( k \)-regular graph on \( n \) vertices is also a counterexample. We have that the average degree in \( G \) is \( k \), and

\[
\text{avg degree in } G - v = \frac{1}{n-1} \left( nk - 2k \right) = \left( \frac{n-2}{n-1} \right) k < k = \text{avg degree in } G
\]
Information you may find helpful:

Special classes of graphs:

- A **path graph** is a graph whose vertices can be labeled \( v_1, v_2, \ldots, v_n \) such that \( v_i \sim v_{i+1} \) for \( 1 \leq i \leq n - 1 \) and no other edges exist in the graph. A path graph on \( n \) vertices is denoted \( P_n \).

  ![Figure 4: The path graphs \( P_1, P_2, P_3, P_4, \) and \( P_5 \).](image)

- A **cycle graph** is a graph with at least 3 vertices whose vertices can be labeled \( v_1, v_2, \ldots, v_n \) such that \( v_i \sim v_{i+1} \) for \( 1 \leq i \leq n - 1 \), \( v_1 \sim v_n \), and no other edges exist in the graph. A cycle graph on \( n \) vertices is denoted \( C_n \).

  ![Figure 5: The cycle graphs \( C_3, C_4, \) and \( C_5 \).](image)

- A **complete graph** is a graph in which every pair of vertices are adjacent. We use \( K_n \) to denote a complete graph on \( n \) vertices.

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1Since we are in the context of simple graphs, the smallest cycle is one with 3 vertices. If, on the other hand, we allowed loops and multiple edges, we could have a cycle graph with one vertex or two vertices.