Matchings in general graphs

We have a characterization of bipartite graphs with perfect matchings (Hall’s theorem). Is there a characterization of general graphs with perfect matchings?

Observation 1 If $M$ is a matching in a graph $G$, each odd component of $G$ must include at least one vertex not saturated by $M$.

Definition An odd component is a component of a graph with an odd number of vertices. We use $o(G)$ to denote the number of odd components of a graph $G$.

So given a matching $M$, the number of unsaturated vertices $\geq o(G)$.

This inequality can be extended by considering induced subgraphs of $G$.

Key idea: for any matching of graph $G$ in Figure 2, at least one vertex in each odd component of $G - v$ must be matched to $v$. So at least two
vertices in $G$ are unsaturated by any matching.
(Note that $o(G - v) = 3 > 1 = |\{v\}|$.)

**Observation 2** If $M$ is a matching in a graph $G$ and $S \subseteq V(G)$, then any odd component in $G - S$ must have at least one vertex matched to a vertex in $S$, i.e.,

the number of unsaturated vertices $\geq o(G - S) - |S|$.

**Theorem** (Tutte). A graph $G$ has a perfect matching if and only if

$$\forall S \subseteq V(G), \quad o(G - S) \leq |S|.$$  

*Tutte’s condition*

**Proof.** ($\Rightarrow$) [Easy.] Suppose $G$ has a perfect matching $M$. Let $S \subseteq V(G)$, and let $G_1, G_2, \ldots, G_t$ be the odd components of $G - S$. Then there must be at least one edge in $M$ from each $G_i$ to $S$, so $|S| \geq t = o(G - S)$.

($\Leftarrow$) [Hard.] Suppose $o(G - S) \leq |S|$ for all $S \subseteq V(G)$, but, for sake of contradiction, assume that $G$ has no perfect matching. In fact, consider such a counterexample $G$ with the maximum $\#$ of edges.

The special case of $S = \emptyset$ in Tutte’s condition implies that $G$ has even $\#$ of vtc$s$ in every component. Note that a complete graph on an even $\#$ of vtc$s$ has a perfect matching, so $G$ is not $K_n$, i.e., has less than $\binom{n}{2}$ edges.

**Strategy:** Show that, in fact, there is perfect matching in $G$.

Let $U = \{u \in V(G) : d(u) = n - 1\}$ where $n = |V(G)|$. (Since $G \neq K_n$, $U \neq V$.) We consider two cases.
**Case 1: Every component of \( G - U \) is a complete graph.**

Choose arbitrary perfect matchings of components of \( G - U \) (which are complete graphs), with odd components each having one vertex that is not yet saturated. Since \( o(G - U) \leq |U| \) and each vertex of \( U \) is adjacent to all vtc's of \( G - U \), we can match leftover vtc's of \( G - U \) to vtc's of \( U \).

Remaining vtc's of \( U \) that are unsaturated form a clique. We claim that there are an even # of such vtc's. We know \( G \) has an even # of vertices total. Furthermore, we have saturated an even # of vtc's, so there must be an even # of vtc's remaining. Thus, we can construct a perfect matching of \( G \). \( \Rightarrow \Leftarrow \)

**Case 2: \( G - U \) has a component that is not a complete graph.**

There must be a pair of vtc's in this component that are dist 2 apart, i.e. \( \exists \) vtc's \( x, y, z \) with \( x \sim y \sim z \) but \( x \not\sim z \).

Since \( y \not\in U \), we have \( d(y) < n - 1 \), so there exists a vertex \( w \) such that \( y \not\sim w \). (This means \( d(w) < n - 1 \) so \( w \not\in U \).)
Recall our previous assumption that $G$ is a “maximum-edge-counter-example.” Note that Tutte’s condition is preserved under addition of edges. So adding any edge to $G$ must yield a graph that has a perfect matching. (Otherwise, we have a larger counterexample than $G$.)

Let

\[ M_1 = \text{perfect matching in } G + xz \]
\[ M_2 = \text{perfect matching in } G + yw \]
\[ H = (V, M_1 \triangle M_2). \]

Since $M_1, M_2$ are perfect matchings, all vtcs of $H$ have degree 0 or 2, and so all components of $H$ are even cycles and isolated vertices.

Need to find matching based on $M_1, M_2$ that does not use edges $xz$ and $yw$ but saturates the same vertices as $M_1, M_2$.

**Case 2(a): $xz$ and $yw$ are in different components of $H$.**

Let $C$ be cycle of $H$ containing $yw$ (edge of $M_2$). Construct $M$ as

\[ M = \{ M_1 \text{ edges of } C \} \]
\[ \cup \{ \text{all } M_2 \text{ edges not in } C \}. \]

Then $M$ is a perfect matching of $H$ and hence of $G$. \( \Rightarrow \Leftarrow \)
Case 2(b): *xz and yw are in same component of H.*

WLOG, assume that \( x/z \) are labeled so that \( x, y, w, z \) appear in that cyclic order on the cycle. Let

\[
M = \{M_1 \text{ edges on } (y, z)-\text{path via } w\} \\
\cup \{\text{all } M_2 \text{ edges not on that path}\} \cup \{yz\}.
\]

Then \( M \) is a perfect matching of \( H \) and hence of \( G \). \( \Rightarrow \Leftarrow \)

Tutte’s theorem can also be proved via Hall’s theorem.

Bottom-line:

- To show that \( G \) has a perfect matching, we exhibit one.
- To show that \( G \) does not have a perfect matching, we find a set \( S \) such that \( G - S \) has too many odd components.

Often Tutte’s theorem is used to show that a graph with some other condition that implies Tutte’s condition has a perfect matching.
Corollary (Petersen). Every 3-regular graph with no cut edge has a perfect matching.

Originally proved prior to Tutte’s theorem.

Outdated but nice language: Every cubic bridgeless graph has a perfect matching.

Proof. Let $G = (V, E)$ be a 3-regular graph without a cut edge, and consider any $S \subseteq V$. We want to show that $o(G - S) \leq |S|$.

Let $H_1, \ldots, H_t$ be odd components of $G - S$, and let $m_i$ be # of edges from $S$ to $H_i$. Then

$$
\sum_{v \in V(H_i)} d(v) = \frac{3|V(H_i)|}{\text{odd}} \quad \text{since } G \text{ is 3-regular}
$$

$$
\sum_{v \in V(H_i)} d(v) = m_i + 2|E(H_i)| \implies m_i = \sum_{v \in V(H_i)} d(v) - 2|E(H_i)|.
$$

Since $G$ has no cut edge, $m_i \neq 1$. Thus, since $m_i$ is odd, $m_i \geq 3$. It follows that

$$
o(G - S) = t = \sum_{i=1}^{t} 1 \leq \frac{1}{3} \sum_{i=1}^{t} m_i
$$

$$
\leq \frac{1}{3} \sum_{v \in S} d(v)
$$

$$
= |S|.
$$

$\blacksquare$
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Odd components of $G - U$

Even components of $G - U$

$v$ts. of degree $n - 1$

even # of vts.