Connectivity and cuts

Math 104, Graph Theory

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Measure of connectivity

How connected are each of these graphs?

—–> increasing connectivity —–>

- $G_1$ is a tree, so it is a connected graph w/minimum # of edges. Every edge is a cut edge.

- $G_2$ has no cut edge but can be disconnected by deletion of one vertex. (Vertex of deg 4 is a cut vertex.)

- $G_3$ has no cut edges and no cut vtc's, but nonetheless, it is not as well connected as $G_4$, a complete graph.
Measure of connectivity

How connected are each of these graphs?

We would like to have a measure of connectivity of a graph.

We consider two today: one in terms of vertices \( \kappa(G) \) and one in terms of edges \( \kappa'(G) \).

Note that the connectivity of a graph gives an indication of its robustness as a network.

Cut sets and disconnecting sets

**Definition**  A set of vertices \( S \) is a **cut set** if \( G - S \) is disconnected.

A set of edges \( F \) is a **disconnecting set** of edges if \( G - F \) is disconnected.

**Example**  Find a cut set and a disconnecting set in the graph below.

Definition

*connectivity* of \( G \): \( \kappa(G) = \) minimum size of a cut set

\( G \) is **\( k \)-connected** if its connectivity is at least \( k \)

*edge connectivity* of \( G \): \( \kappa'(G) = \) minimum size of a disconnecting set

\( G \) is **\( k \)-edge-connected** if its edge connectivity is at least \( k \)
Questions

- When is \( \kappa(G) = 0 \)? When is \( \kappa'(G) = 0 \)?
- When is \( \kappa(G) = 1 \)? When is \( \kappa'(G) = 1 \)?
- What is an example of a graph \( G \) for which \( \kappa(G) < \kappa'(G) \)?
- What is an example of a graph \( G \) for which \( \kappa'(G) < \kappa(G) \)?

An example

**Example** For our previous example with graph \( G \) below

we have \( \kappa(G) = 1 \) since
- \( G \) is connected (which means \( \kappa(G) > 0 \))
- and \( G \) has a cut vertex (so \( \kappa(G) \leq 1 \)).

Also, \( \kappa'(G) = 1 \) since \( G \) is a connected graph with a cut edge.
Are $\kappa(G)$ and $\kappa'(G)$ well-defined for all graphs?

What is $\kappa(K_n)$? \(\implies\) No matter how many vtcs we remove from the complete graph, we never disconnect it.

We tweak our definition of connectivity to handle this case.

That is, the connectivity of $G$, denoted $\kappa(G)$, is the minimum size of a set $S \subseteq V(G)$ such that either $G - S$ is disconnected or $G - S$ has one vertex.

Thus, \[ \kappa(K_n) = n - 1. \]

Are $\kappa(G)$ and $\kappa'(G)$ well-defined for all graphs?

Along the same lines as our last remark, are there any graphs $G$ for which $\kappa'(G)$ is not well-defined?

How about $\kappa'(K_1)$? $K_1$ is connected and there are no edges to remove to try and disconnect the graph.

There is no good definition for this value, so we will just define it to be zero, i.e. $\kappa'(K_1) = 0$. 
More examples to consider

- What is \( \kappa(K_{m,n}) \)?
- The \( d \)-dimensional hypercube \( Q_d \) is the simple graph whose vertices are the binary sequences (all entries are 0, 1) of length \( d \) and two binary sequences are adjacent if and only if they differ in exactly one position.

For example, two drawings of \( Q_3 \) are shown below:

![Diagram of \( Q_3 \)]

What is \( \kappa(Q_d) \)?

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Edge cuts

**Definition** Given disjoint (nonempty) sets \( S, T \subseteq V(G) \) for a graph \( G \), \( [S, T] \) denotes the set of edges with one endpoint in \( S \) and one endpoint in \( T \),

\[
[S, T] = \{ uv \in E(G) : u \in S, v \in T \}.
\]

In particular, we are often interested in the case where \( T = \overline{S} = V(G) - S \). The set of edges in \( [S, \overline{S}] \) is called an edge cut of \( G \).

**Example**

![Diagram of edge cut](image)

Yellow vtcs are vtcs in \( S \), black vtcs are those in \( \overline{S} \), and edges crossed by dashed red line are edges in the edge cut \( [S, \overline{S}] \).
Edge cuts

**Question** What is the relationship between a disconnecting set of edges in $G$ and an edge cut set of $G$? In other words...

- Is an edge cut always a disconnecting set?
  
  $\implies$ **Yes!** No path exists from vertex in $S$ to vertex in $\overline{S}$.

- Is a disconnecting set always an edge cut?
  
  $\implies$ A **minimal** disconnecting set is an edge cut.

So we can adjust our definition of $\kappa'(G)$, if we like:

$$\kappa'(G) = \text{minimum size of disconnecting set}$$

$$\text{edge cut}$$

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Whitney’s theorem

What is the relationship between connectivity, edge connectivity, and the degrees of vtc's in a graph?

**Theorem (Whitney)**

*Let $G$ be a graph. Then*

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

Before we prove this theorem, we mention one basic result and leave the proof as an exercise.

**Lemma**

*Let $G$ be a connected graph with at least three vertices. If $G$ has a cut edge $e = uv$, then at least one of $u, v$ is a cut vertex.*
Proof of Whitney’s inequality: $\kappa'(G) \leq \delta(G)$

The result $\kappa(G) \leq \kappa'(G) \leq \delta(G)$ is trivially true for graph $K_1$, so assume $G$ has at least two vtcs.

We first show that $\kappa'(G) \leq \delta(G)$. Let $v$ be a vertex of minimum degree.

![Diagram of a vertex v with edges to its neighbors in V(G)-{v}]

Note that $[\{v\}, V(G) - \{v\}]$ is an edge cut of size $\delta(G)$.

Thus, $\kappa'(G) \leq \delta(G)$.

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Proof of Whitney’s inequality: $\kappa(G) \leq \kappa'(G)$

Next we show that $\kappa(G) \leq \kappa'(G)$ via a proof by induction on $\kappa'(G)$.

**Base case:** If $\kappa'(G) = 0$, then $G$ is disconnected so $\kappa(G) = 0$.

**Induction hypothesis:** Suppose that for any graph $G$ with $\kappa'(G) = k$, we have $\kappa(G) \leq \kappa'(G)$.

**Inductive step:** Let $G$ be a graph with $\kappa'(G) = k + 1$, and let $F \subseteq E(G)$ be an edge cut of size $k + 1$. Suppose $e \in F$, and let $H = G - e$.

Then $F - e$ is a minimum edge cut of $H$, so $\kappa'(H) = k$. Apply the induction hypothesis to $H$ to conclude that $\kappa(H) \leq \kappa'(H) = k$. 
Proof of Whitney’s inequality: $\kappa(G) \leq \kappa'(G)$

Let $S$ be an optimal (vertex) cut set of $H$. Then $H - S$ is disconnected. We consider two cases:

Case 1: $G - S$ is disconnected.
Then
$$\kappa(G) \leq |S| = \kappa(H) \leq k < \kappa'(G).$$

Case 2: $G - S$ is connected. Since $G - S$ is connected but $H - S$ is not, we know that $e$ is a cut edge of $G - S$.

Since $e$ has endpoints in $V(G) - S$, we have $|V(G) - S| \geq 2$.

- If $|V(G) - S| = 2$, then $|S| = |V(G)| - 2$, so
$$\kappa(G) \leq |V(G)| - 1 = |S| + 1 = \kappa(H) + 1 \leq k + 1 = \kappa'(G).$$

- If $|V(G) - S| > 2$, then an endpoint of the cut edge $e$ is a cut vertex of $G - S$ (by previous lemma). Let $v$ be such a cut vertex. Then $S \cup \{v\}$ is a cut set for $G$, so
$$\kappa(G) \leq |S| + 1 = \kappa(H) + 1 \leq k + 1 = \kappa'(G).$$
Testing the bounds of Whitney’s theorem

Find examples of a graph $G$ such that

- $\kappa(G) < \kappa'(G) < \delta(G)$
- $\kappa(G) = \kappa'(G) < \delta(G)$
- $\kappa'(G) < \kappa'(G) = \delta(G)$. 