Chromatic polynomials (continued)

Recall two results we saw last class.

**Theorem** (deletion/contraction formula). *Let* $e$ *be an edge of a simple graph* $G$. *Then*

\[ \chi(G; k) = \chi(G - e; k) - \chi(G \cdot e; k). \]

**Theorem.** *Suppose* $G$ *is a disconnected graph with components* $H_1, H_2, \ldots, H_c$. *Then*

\[ \chi(G; k) = \chi(H_1; k) \cdot \chi(H_2; k) \cdots \chi(H_c; k). \]

**Proof.** Every $k$-coloring of $G$ is formed by independently finding a $k$-coloring of each of its components.

**Example: Kite graph**  Using deletion/contraction formula, we have

\[
\chi(K_4 - e; k) = \chi(K_1; k)\chi(K_3; k) - 2\chi(K_3; k)
\]

\[
= k[k(k - 1)(k - 2)] - 2k(k - 1)(k - 2)
\]

\[
= \frac{k(k - 1)(k - 2)^2}{k(k - 1)(k - 2)} = k^4 - 5k^3 + 8k^2 - 4k.
\]
Was an easier approach possible?

$\implies$ add an edge to obtain a graph whose chromatic polynomial is easy to determine (in this case, $K_4$)

**Observations about the chromatic polynomial**

- Polynomial.
- Terms have alternating signs.
- Degree of polynomial = # of vtc's in $G$.
- Leading coefficient is 1 (monic polynomial).
- All coefficients are integers.
- Next-below-leading coefficient = -# of edges in $G$.
- Constant term is 0.

**Theorem.** Let $G$ be a graph with $n$ vertices and $m$ edges. Then $\chi(G; k)$ is a monic, integer polynomial in $k$ of degree $n$ with constant term 0. The signs of this polynomial alternate and the coefficient of $k^{n-1}$ is $-m$.

**Proof.** Proof is by strong induction $m = |E(G)|$.

*Base case:* If $m = 0$, then $G$ is an edgeless graph and $\chi(G; k) = k^n$, which satisfies the theorem.

*Strong induction hypothesis:* Suppose the theorem is true for all graphs with at most $m$ edges.
Let $G$ be a graph with $m + 1$ edges, and let $e \in E(G)$. Note that $G - e$ has $m$ edges and $G \cdot e$ has at most $m$ edges. We apply the induction hypothesis to $G - e$ and $G \cdot e$ to obtain

\[
\chi(G - e; k) = k^n - mk^{n-1} + \cdots - \cdots + \cdots \pm 0
\]

\[-[ \chi(G \cdot e; k) = k^{n-1} - \cdots + \cdots - \cdots \mp 0 ]
\]

\[
\chi(G; k) = k^n - (m + 1)k^{n-1} + \cdots - \cdots + \cdots \pm 0
\]

Therefore, by strong induction, the result holds.

\[\blacksquare\]

**Proposition.** For any graph $G$, $\chi(G; k)$ may be computed iteratively using the deletion-contraction formula. Furthermore, $\chi(G; k)$ is a polynomial in $k$ with roots $0, 1, 2, \ldots, \chi(G) - 1$ and no other positive integers as roots.

Iterating deletion/contraction yields $2^{|E(G)|}$ terms as we dispose of all edges.

**Other remarks on chromatic polynomials**

- Nonisomorphic graphs may have the same chromatic polynomial. See example below.
A graph that is determined by its chromatic polynomial is said to be a *chromatically unique graph*. Nonisomorphic graphs sharing the same chromatic polynomial are said to be *chromatically equivalent*.

- Open questions:
  
  - Which polynomials are the chromatic polynomials of some graph? (Read, 1968)
  
  - Birkhoff-Lewis conjecture (1946): The chromatic polynomial of a planar graph has no roots in \([4, \infty)\). (Proven that there are no real roots in \([5, \infty)\). Still open for interval \((4, 5)\).

**Proposition.** For any graph \(G = (V, E)\) and positive integer \(i\), let \(p_i(G)\) be the \# of ways to partition \(V\) into exactly \(i\) (nonempty) independent sets. Then

\[
\chi(G; k) = \sum_{i=1}^{\lvert V \rvert} p_i(G)(k)_i = \sum_{i=1}^{\lvert V \rvert} p_i(G) \frac{k!}{(k - i)!}.
\]

**Question**  Technically, we can calculate \(\chi(G; -1)\) since \(\chi(G; k)\) is a polynomial. But does this make any sense?
Counting proper colorings

\[ \chi(G; k) = k^8 - 12k^7 + 66k^6 - 214k^5 + 441k^4 - 572k^3 + 423k^2 - 133k \]

\[ \chi(G; 3) = 114 \]
Combinatorial reciprocity
(a last look at chromatic polynomials)

Combinatorial reciprocity (Stanley, 1974) the relationship between a combinatorially-defined polynomial (defined over nonnegative integers) and its extension to negative integers

...basic idea – the polynomial over negative integers also has a combinatorial meaning but one that might possibly be quite different than that of the original polynomial.

Example  Let $i > 0$ be a fixed value. Consider

$$\binom{k}{i} = \frac{\text{# of ways of choosing } i \text{ objects from } k \text{ objects (without repetition permitted)}}{i!}$$

So we view this as a polynomial in $k$.

Then

$$\binom{-k}{i} = \frac{-k(-k-1)(-k-2)\cdots(-k-i+1)}{i!}$$

$$= (-1)^i \frac{k(k+1)(k+2)\cdots(k+i-1)}{i!}$$

$$= (-1)^i \frac{(k+i-1)!}{(k-1)! \cdot i!}$$

$$= (-1)^i \left( \binom{k+i-1}{i} \right) = (-1)^i \left( \binom{k}{i} \right).$$

$$\implies \left( \binom{k}{i} \right) = (-1)^i \left( \binom{-k}{i} \right)$$

What about $\chi(G; -1)$?
**Theorem.** For a simple graph $G$ on $n$ vertices,

$$(-1)^n \chi(G; -1)$$

is the # of ways to orient (assign directions to) the edges of $G$ so that the digraph has no directed cycles.

$$(-1)^n \chi(G; -1) = \# \text{ of acyclic orientations of } G$$

**Examples**

- **Paw graph $G$**

  $\chi(G; k) = \chi(G - e; k) - \chi(G \cdot e; k)$

  $$= k^2(k - 1)(k - 2) - k(k - 1)(k - 2)$$

  $$= k(k - 1)^2(k - 2)$$

  so we have

  $$(-1)^4 \chi(G; -1) = -1(-2)^2(-3) = 12.$$ There are $2^4 = 16$ possible orientations of edges of $G$; only 4 orientations have a directed cycle:

- **Tree $T$ on $n$ vtds**

  $$\chi(T; k) = k(k - 1)^{n-1}$$

  $$(-1)^n \chi(T; -1) = (-1)^n(-1)(-2)^{n-1} = 2^{n-1},$$

  so ALL orientations of $T$ are acyclic!
Proof of theorem. Let $a(G)$ denote the # of acyclic orientations of $G$. We give a proof by induction on $m = |E(G)|$. Assume $G$ has $n$ vtds.

Base case: $m = 0$. $G$ is edgeless and has one acyclic orientation (the empty one), so

$$a(G) = 1 = (-1)^n(-1)^n = (-1)^n \chi(G; -1)$$

since $\chi(G; k) = k^n$.

IH: Assume the result holds for any graph with $m$ edges.

Let $G$ be a graph with $m + 1$ edges.

Claim: $a(G) = a(G - e) + a(G \cdot e)$ for any edge $e \in E(G)$.

Then

$$a(G) = a(G - e) + a(G \cdot e) \quad \text{by above claim}$$

$$= (-1)^n \chi(G - e; -1) + (-1)^{n-1} \chi(G \cdot e; -1) \quad \text{by IH}$$

$$= (-1)^n \left[ \chi(G - e; -1) - \chi(G \cdot e; -1) \right] \quad \text{by factoring}$$

$$= (-1)^n \chi(G; -1) \quad \text{by deletion/contraction formula for } \chi(G; k).$$

Proof of Claim. Consider how we may extend an acyclic orientation (a.o.) $D$ of $G - e$ to an a.o. of $G$. Let $e = \{u, v\}$.

We claim any a.o. of $G - e$ will extend to 1 or 2 a.o.’s of $G$.

Since $D$ is acyclic, there cannot be both a $u, v$-path and a $v, u$-path in $D$.

- If $D$ has no $u, v$-path, orient edge $e$ as $(v, u)$.
- If $D$ has no $v, u$-path, orient edge $e$ as $(u, v)$. 
So
\[
a(G) = (\text{# of a.o. of } G - e) \\
+ (\text{# of a.o. of } G - e \text{ with no } u, v\text{-path and no } v, u\text{-path}) \\
= a(G - e) + a(G \cdot e).
\]

Note that a \(u, v\)-path or a \(v, u\)-path in orientation of \(G - e\) is a cycle in corresponding orientation of \(G \cdot e\).

\[\]

**Theorem** (Reciprocity of chromatic polynomial). \(^1\) Let \(G\) be a graph with \(n\) vtc.

Let \(D\) be an acyclic orientation of \(G\). For a not necessarily proper coloring \(f\) of \(G\) using colors \(\{1, 2, \ldots, k\}\), we say \((D, f)\) is a compatible pair if for each arc \((u, v)\) in \(D\), we have \(f(u) \leq f(v)\).

Then the \# of compatible pairs for \(G\) is

\[
(-1)^n \chi(G; -k).
\]

**Exercise** Put this result into action with graph \(P_3\) and \(k = 2\).

\[
\chi(P_3; k) = k(k - 1)^2
\]

\[
(-1)^3 \chi(P_3; -2) \\
= -(-2)(-3)^2 \\
= 18
\]

\[\]

\(^1\)This result is found in your textbook in Problem 5.3.32 but there is a typo. It should read \(\eta(G; k) = (-1)^n(G)\chi(G; -k)\).
Remark  Notice that our previous theorem is just a special case \((k = 1)\) of this more general result. There is

- only one 1-coloring of any graph \(G\) (assign color 1 to all vtc's),
- and every acyclic orientation is compatible with this coloring,

so the \# of compatible pairs is simply the \# of acyclic orientations of \(G\).