Planarity: coloring and higher genus surfaces

Math 104, Graph Theory

April 2, 2013

Coloring planar graphs
Six Color Theorem

Theorem

Let $G$ be a planar graph. Then $\chi(G) \leq 6$.

Proof highlights.

- **Proof type:** Induction on $|V(G)|$.
- **Basic argument:** Color min degree vertex with a leftover color.
- **Key fact:** $\delta(G) \leq 5$.

Five Color Theorem

Theorem

Let $G$ be a planar graph. Then $\chi(G) \leq 5$.

Proof highlights.

- **Proof type:** Induction on $|V(G)|$.
- **Basic argument:** Color min degree vertex $v$ with a leftover color, if possible. Otherwise, swap two colors to make a color available at $v$.
- **Key facts:**
  - $\delta(G) \leq 5$.
  - Cannot have both a 1-3 alternating path and a 2-4 alternating path.
Five Color Theorem

Theorem

Let $G$ be a planar graph. Then $\chi(G) \leq 5$.

Proof highlights.

- **Proof type:** Induction on $|V(G)|$.

- **Basic argument:** Color min degree vertex $v$ with a leftover color, if possible. Otherwise, swap two colors to make a color available at $v$.

- **Key facts:**
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  - Cannot have both a 1-3 alternating path and a 2-4 alternating path.

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on component of $G[2/4 vtc]$ containing $w$, swap all 2/4 colored $vtc$
Four Color Theorem

Theorem
Let $G$ be a planar graph. Then $\chi(G) \leq 4$.

A flawed proof.

Proof by induction on $|V(G)|$.

Base case: Any graph $G$ with $|V(G)| \leq 4$ is trivially 4-colorable.

Induction hypothesis: Suppose any planar graph with less than $n$ vtc's is 4-colorable.

Let $G$ be a planar graph w/ $n$ vtc's. If $G$ is not maximal planar, then add edges to make it so. Consider a planar embedding of $G$. 
Four Color Theorem

Proof continued.

Let $v$ be a min degree vertex.

- If $d(v) \leq 3$ or if there are at most 3 colors on neighbors of $v$, apply a leftover color argument.

- If $d(v) = 4$ and all 4 colors are present on neighbors of $v$, use a Kempe chain argument (as in proof of Five Color Thm).

The hard case is left: $d(v) = 5$ and all 4 colors are present on neighbors (so some color is repeated exactly once).

Since $G$ is maximal planar, its embedding is a triangulation, so $N(v)$ form a 5-cycle. WLOG, assume colors of $N(v)$ are 1, 2, 3, 4, 2 cyclically.

Proof continued.

- If we can swap colors 1/3 (on comp of $G[1/3]$ containing $u$), do it; then color $v$ w/color 1. Otherwise, $\exists$ a 1/3 Kempe chain.

- If we can swap colors 1/4 (on comp of $G[1/4]$ containing $u$), do it; then color $v$ w/color 1. Otherwise, $\exists$ a 1/4 Kempe chain.
**Four Color Theorem**

Proof continued.

1. If we can swap colors 1/3 (on comp of $G[1/3]$ containing $u$), do it; then color $v$ w/color 1. Otherwise, $\exists$ a 1/3 Kempe chain.

2. If we can swap colors 1/4 (on comp of $G[1/4]$ containing $u$), do it; then color $v$ w/color 1. Otherwise, $\exists$ a 1/4 Kempe chain.


Thus, we have a proper 4-coloring of $G$ and the result holds by induction.

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**What’s wrong with this proof?**

While this proof by Kempe is wrong, it stood unchallenged for over 10 years. (Though flawed as a proof for the Four Color Thm, the ideas were used in a proof by Heawood of the Five Color Thm.)

**Problem:** The 1/3 Kempe path and 1/4 Kempe path may cross (at a vertex of color 1). Swapping 2/3 and 2/4 then causes two adjacent vtc's to be assigned color 2.
An actual proof of the Four Color Theorem

A proof was eventually given by Appel and Haken in the mid-1970’s that showed that a minimal counterexample could not exist; they used ideas of *unavoidable sets* and *reducible configurations*.

Part of the Appel-Haken proof consisted of an exhaustive analysis of many discrete cases (originally 1936) by a computer; controversy was stirred because the proof was computer-assisted.

Robertson, Sanders, Seymour, and Thomas improved Appel and Haken’s approach by reducing the number of cases to be checked by computer to 633.

633 cases
Surfaces of higher genus

Handles and genus

We consider (closed, orientable) surfaces that can be constructed from the sphere as follows:

- cut out 2 disjoint discs of equal radius from sphere
- bend and attach a cylinder of the same radius to the sphere so that the ends of the cylinder are glued to the boundaries of the discs

adding a handle
Holes in a surface

**Definition**  The *genus* of a surface is the number of handles added to a sphere to obtain the surface.

Formally speaking, the genus is the largest # of nonintersecting simple closed curves that can be drawn on the surface without disconnecting it. Roughly speaking, it is the # of holes in a surface.

**Remark**  Two surfaces are considered the same if one can be continuously deformed into the other.

"A topologist is a mathematician who cannot tell the difference between a coffee mug and a donut—both are surfaces of genus 1." – MathWorld

This animation shows a continuous deformation, known as a *homeomorphism*, of a coffee mug into a donut and then back to a coffee mug. This surface is a torus.

<table>
<thead>
<tr>
<th>genus 0</th>
<th>genus 1</th>
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<tr>
<td>[Image]</td>
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<table>
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<tr>
<th>genus 2</th>
<th>genus 3</th>
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<td>[Image]</td>
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Euler characteristic – initial definition

**Definition**  The *Euler characteristic* of a surface \( \Sigma \) is

\[
 n - e + f
\]

where

\[
 n = |V(G)|,
 e = |E(G)|,
 and \( f = \# \text{ of faces} \)

in an embedding of a connected graph \( G \) on surface \( \Sigma \).

---

Euler characteristic

**Claim:** The Euler characteristic is an *invariant* of the surface, (constant value, independent of choice of connected graph or its embedding).

Let's try a simple example – a graph \( G \) consisting of one vertex and two loops. How many faces are there in an embedding of \( G \) on the torus?

1, 2, 3 ?
2-cell embeddings

**Definition** A 2-cell is a region such that every closed curve in the interior can be continuously contracted to a point.

*Alternatively, it is a region homeomorphic to an open disc.*

**Definition** A 2-cell embedding is a crossing-free drawing of a graph on a surface where every face is a 2-cell.

**Euler characteristic – updated definition**

**Definition** The Euler characteristic of a surface $\Sigma$ is

$$n - e + f$$

where

- $n = |V(G)|$,
- $e = |E(G)|$,
- and $f = \# \text{ of faces}$

in a 2-cell embedding of a connected graph $G$ on surface $\Sigma$. 
Value of Euler characteristic for surfaces

**Theorem**
*If G has a 2-cell embedding on a surface of genus g, then*

\[ n - e + f = 2 - 2g. \]

So each time we add a handle to a surface, the Euler characteristic decreases by 2.

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**Seven Color Theorem**

**Theorem**
*If G is toroidal, then* \( \chi(G) \leq 7. \)

A map on torus that requires 7 colors. (The map corresponds to \( K_7 \).)
Seven Color Theorem

Theorem

*If G is toroidal, then* \( \chi(G) \leq 7 \).

A map on torus that requires 7 colors. (The map corresponds to \( K_7 \).)
Genus of a graph

**Definition**  The *genus* of a graph $G$ is the smallest genus on which $G$ has a 2-cell embedding and is denoted $\gamma(G)$.

**Examples**
- $\gamma(K_{1,3}) = 0$ since $K_{1,3}$ is planar
- $\gamma(K_5) = 1$ since $K_5$ is toroidal but not planar
- $\gamma(K_6) = 1$ since $K_6$ is toroidal but not planar