Hamiltonian graphs (continued)

Is the Grötzsch graph Hamiltonian?

Warmup problem

Review: Deciding whether a given graph is Hamiltonian is NP-complete, so it is natural to look for necessary or sufficient conditions for the existence of Hamiltonian cycles. Some examples we saw last class:

- components condition: \( c(G - S) \leq |S| \) (necessary)
- min degree large enough: \( \delta(G) \geq n/2 \) (sufficient)
- every pair of nonadjacent vtc\( s \) has degree sum \( \geq n \) (sufficient)
- if some pair of nonadjacent vtc\( s \) \( u, v \) have degree sum \( \geq n \), then \( G \) is Hamiltonian if and only if \( G + uv \) is Hamiltonian (necessary and sufficient, in a special case)

On a chessboard, a knight can move from one square to another that differs by 1 in one coordinate and by 2 in the other coordinate, as shown below. Prove that no \( 4 \times n \) chessboard has a knight’s tour: a traversal by knight’s moves that visits each square once and returns to the start.

Problem

On a chessboard, a knight can move from one square to another that differs by 1 in one coordinate and by 2 in the other coordinate, as shown below. Prove that no \( 4 \times n \) chessboard has a knight’s tour: a traversal by knight’s moves that visits each square once and returns to the start.
Solution  Note that every neighbor of a white square in the top and bottom rows is a black square in the middle two rows. Therefore, if we delete the $n$ black squares in the middle two rows, the white squares in the top and bottom rows become $n$ isolated vtc's.

There remain $2n$ other vtc's in the graph, which must form at least one more component. Hence we have found a set $S$ of $n$ vtc's whose deletion leaves at least $n + 1$ components, which means that $G$ cannot be Hamiltonian.

Definition  The closure of a graph $G$, denoted $C(G)$, is defined recursively as follows:

- If $G$ has no pair of nonadjacent vtc's $u, v$ such that $d(u) + d(v) \geq n$, then set $C(G) = G$.
- Otherwise, let $u, v$ be a nonadjacent pair of vtc's with $d(u) + d(v) \geq n$ and set $C(G) = C(G + uv)$.

Example

Does $C(G)$, for a given graph $G$, depend on the order in which we add edges? (In other words, is $C(G)$ well-defined?)
**Lemma.** For a graph $G$, $C(G)$ does not depend on the order in which we choose to add edges when more than one is available.

**Proof.** Suppose $G_1$ and $G_2$ are obtained as $C(G)$ from $G$ by two different implementations of the closure procedure. Let $n = |V(G)|$.

Let $e_1, e_2, \ldots, e_s$ and $f_1, f_2, \ldots, f_t$ denote the sequences of edges added to $G$ to make $G_1$ and $G_2$, respectively.

**Claim:** Every edge of $G_1$ is in $G_2$ and vice-versa, i.e., $e_i \in E(G_2)$ and $f_j \in E(G_1)$ $\forall i, j$.

Suppose not. Let $e_{k+1} = uv$ be the first edge of $G_1$ not in $G_2$. Consider graph $H$ obtained from $G$ by adding edges $e_1, e_2, \ldots, e_k$. Then

- $e_{k+1} \in E(G_1)$ implies that $d_H(u) + d_H(v) \geq n$,
- and $H$ is a subgraph of $G_2$, so $d_{G_2}(u) \geq d_H(u)$ and $d_{G_2}(v) \geq d_H(v)$.

It follows that $d_{G_2}(u) + d_{G_2}(v) \geq d_H(u) + d_H(v) \geq n$. Thus, $e_{k+1}$ should be an edge of $G_2$. $\Rightarrow\Leftarrow$

Therefore, $E(G_1) = E(G_2)$ and since $G$ is a spanning subgraph of both $G_1$ and $G_2$, we have that $G_1 = G_2$ and $C(G)$ is well-defined. $\blacksquare$

**Theorem** (Bondy-Chvátal). Let $G$ be a graph with $n \geq 3$ vtc$. Then $G$ is Hamiltonian if and only if $C(G)$ is Hamiltonian.
**Corollary.** Let $G$ be a graph with $n \geq 3$ vtcs. If $C(G)$ is a complete graph, then $G$ is Hamiltonian.

This corollary can be used to derive other sufficient conditions for a graph to be Hamiltonian. For example, Chvátal extended Dirac’s theorem to a wider class of graphs (and the proof uses notion of closure of a graph).

**Theorem (Chvátal).** Let $G$ be a graph with $n \geq 3$ vtcs and degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If there is no integer $k < \frac{n}{2}$ such that $d_k \leq k$ and $d_{n-k} < n - k$, then $G$ is Hamiltonian.

Intuition: some vertex degrees can be small if others are large enough to compensate.
Long paths in digraphs

Theorem. Let $D$ be a directed graph. Then $D$ has a directed path of length $\chi(D) - 1$.

Remark. Note that the chromatic # of a directed graph is simply the chromatic # of the underlying undirected graph.

Proof. Let $A$ be a minimum set of arcs so that $D' = D - A$ has no directed cycles. Let $k$ be the length of a longest directed path in $D'$.

For each vertex $v$ in $D'$, if $i$ is the length of a longest path that ends at $v$, color vertex $v$ with color $i + 1$. This coloring uses $k + 1$ colors.

Claim: For any directed path $P$ in $D'$, the colors of the vertices in $P$ are strictly increasing along the path.

To see this, let $u, v$ be endpoints of $P$. Then any directed path that ends at $u$ has no other vertex on $P$ since $D'$ is acyclic. Therefore, any path ending at $u$ (including the longest one) can be extended by concatenating it with $P$. Thus, the colors increase along the path $P$.

This coloring is a proper coloring since for any arc $(u, v)$ in $E(D)$, there is a directed path between $u$ and $v$:

- either $(u, v)$ in $E(D')$
- or there is a directed path from $v$ to $u$ in $D'$.

This implies that $u$ and $v$ cannot be assigned the same colors, since colors increase along any directed path in $D'$. 
Therefore, \( \chi(D) \leq k + 1 \), or \( k \geq \chi(D) - 1 \). So \( D \) has a directed path of length \( \chi(D) - 1 \).

\[ \blacksquare \]

**Example**

This result is best possible.

**Proposition.** Let \( G \) be a graph with \( \chi(G) = k \). Then there exists an orientation \( D \) of \( G \) so that the longest directed path in \( D \) has length \( k - 1 \).