Long paths in digraphs (continued)

Recall we ended last class with the following result and its proof.

**Theorem.** Let $D$ be a directed graph. Then $D$ has a directed path of length $\chi(D) - 1$.

**Example**

This result is best possible.

**Proposition.** Let $G$ be a graph with $\chi(G) = k$. Then there exists an orientation $D$ of $G$ so that the longest directed path in $D$ has length $k - 1$.

**Proof.** Let $f : V(G) \to \{1, 2, \ldots, k\}$ be a proper coloring of $G$. If $uv \in E(G)$, orient the edge as $(u, v)$ if $f(u) < f(v)$; otherwise, orient it as $(v, u)$ if $f(v) < f(u)$.

Every directed path must increase in color, so no path can have length exceeding $k - 1$.

**Example**
Definition  A tournament is an orientation of a complete graph.

Remarks

• Orientations of $K_n$ are often used to model “round-robin tournaments”.

  – Consider a league with $n$ teams where every team plays every other team exactly once. For each pair $u, v$, include the edge $(u, v)$ if $u$ wins or $(v, u)$ if $v$ wins.
  – The “score” of a team is its outdegree (which is its # of wins).
  – Here, we want to determine an overall winning team for the round-robin tournament.

• Tournaments can also be used for modeling “paired comparisons”.

  – An experiment is conducted to rank a number of given objects by comparing only two at a time. Items are presented two at a time to a subject and he is asked to choose his favorite of the two.
  – After having considered all possible pairs of the $n$ objects, the experiment wants to rank the $n$ objects in order of preference.

Figure 1: Results from a classic experiment by researcher named Kendall on six different dog foods. Two foods were served to a dog, and dog established preference by which plate he finished first. Experiment conducted over 15 days.

In the first context, it makes sense to simply rank the teams by their out-degrees (# of wins). However, there may be a tie amongst several
teams for the “top spot”.

Another approach in the same context is to choose a *king* as a winner. In a digraph, a *king* is a vertex from which every other vertex is reachable by a path of length at most 2. It can be shown that every tournament has a king. (For those of you who took Discrete Math w/Prof. Benjamin, this is his discussion of King Chicken Theorems and pecking order.)

In the second context, it may make more sense to rank according to a directed Hamiltonian path. In graph of Figure 1, such a path is 1, 3, 2, 5, 6, 4.

**Theorem.** *Every tournament has a directed Hamiltonian path.*

**Proof.** Since \( \chi(K_n) = n \), by the previous theorem, any orientation of \( K_n \) must have a directed path of length \( n - 1 \). Such a path includes every vertex, so it is a directed Hamiltonian path.

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**Other related results on tournaments**

- **Theorem.** A tournament (with more than 2 vertices) is Hamiltonian if and only if it is strongly connected.

A directed graph is *strongly connected* if there exists a directed path between every ordered pair of vtc's.

- **Theorem.** A tournament has a unique Hamiltonian path if and only if the tournament is transitive.

A tournament is *transitive* if whenever \((a, b)\) and \((b, c)\) are arcs of the tournament, then \((a, c)\) is also an arc.
**Ramsey theory**

At a party of 6 people, there are either three mutual acquaintances or three mutual strangers. For any graph $G$ with at least six vertices, $\omega(G) \geq 3$ or $\alpha(G) \geq 3$.

For any graph $G$ with at least six vertices, $\omega(G) \geq 3$ or $\alpha(G) \geq 3$.

**Remark** A version of this problem appeared on the 1953 Putnam exam.

The complete graph with 6 points and 15 edges has each edge colored red or blue. Show that we can find 3 points such that the 3 edges joining them are the same color.

**Definition** For positive integers $j$ and $k$, the *Ramsey number*, $R(j, k)$ is the smallest integer $n$ such that for all graphs $G$ with $n$ vertices, either $\omega(G) \geq j$ or $\alpha(G) \geq k$.

Equivalent definitions:

- $R(j, k)$ is the smallest integer $n$ such that any red/blue edge-coloring of $K_n$ contains either a red $K_j$ or a blue $K_k$.

- $R(j, k)$ is the min # of guests that have to be invited to a party to guarantee that at least $j$ are acquaintances or $k$ are strangers.

Ramsey proved in 1930 (Ramsey’s theorem) that generalized Ramsey numbers are finite.

General interpretation: every sufficiently large structure, no matter how disorderly it may seem, contains an orderly substructure of any prescribed size.
Easy properties of Ramsey numbers

- Symmetry of Ramsey numbers: \( R(j, k) = R(k, j) \).
- Small values: \( R(j, 1) = 1 \),
  \[ R(j, 2) = j \text{ for } j \geq 2. \]

**Theorem.** Let \( j, k \geq 2 \) be integers. Then

\[ R(j, k) \leq R(j - 1, k) + R(j, k - 1). \]

**Proof.** Let \( n = R(j - 1, k) + R(j, k - 1) \). Consider a red/blue edge-coloring of \( K_n \), and let \( v \) be any vertex.

**Claim:** Either there are at least \( R(j - 1, k) \) red edges at \( v \) or there are at least \( R(j, k - 1) \) blue edges at \( v \).

Suppose not. Then

\[ n - 1 = d(v) \leq R(j - 1, k) - 1 + R(j, k - 1) - 1 = n - 2 \]

which is clearly a contradiction.

**Case 1:** There are at least \( R(j - 1, k) \) red edges at \( v \).

Let \( S \subseteq N(v) \) be the vertices joined to \( v \) by red edges. Since \(|S| \geq R(j - 1, k)\), there must either be a red \( K_{j-1} \) or a blue \( K_k \). In the former case, take the red \( K_{j-1} \) together with \( v \) to obtain a red \( K_j \). (In the latter case, we’re already done.)

**Case 2:** There are at least \( R(j, k - 1) \) blue edges at \( v \).

Analogous to the previous case, we are guaranteed either a red \( K_j \) or a blue \( K_{k-1} \) amongst neighbors of \( v \). Combine \( v \) with blue \( K_{k-1} \) to obtain a blue \( K_k \).
**Corollary.** Let \( j, k \geq 2 \). Then

\[
R(j, k) \leq \binom{j + k - 2}{k - 1}.
\]

**Proof.** We prove by induction on \( j + k \).

**Base cases:** If \( j + k \leq 5 \), then \( \min\{j, k\} \leq 2 \), so by previous statements, we can show that \( R(j, k) \leq \binom{j + k - 2}{k - 1} \). (We may assume furthermore that \( j, k > 2 \).)

**Induction hypothesis:** Suppose the result holds for sums \( j + k = m \).

Assume \( j + k = m + 1 \). Using previous theorem, we have

\[
R(j, k) \leq R(j - 1, k) + R(j, k - 1)
\]

\[
\leq \binom{(j - 1) + k - 2}{k - 1} + \binom{j + (k - 1) - 2}{k - 2} \quad \text{by IH}
\]

\[
= \binom{j + k - 3}{k - 1} + \binom{j + k - 3}{k - 2}
\]

\[
= \binom{j + k - 2}{k - 1} \quad \text{by Pascals identity.}
\]

\[\blacksquare\]

**Remarks**

- In general, exact values of Ramsey numbers are difficult to find.
- \( R(3, 3) = 6 \), \( R(4, 4) = 18 \). Exact value of \( R(5, 5) \) is unknown.

"Imagine an alien force, vastly more powerful than us landing on Earth and demanding the value of \( R(5, 5) \) or they will destroy our planet. In that case, we should marshal all our
computers and all our mathematicians and attempt to find
the value. But suppose, instead, that they asked for $R(6,6)$,
we should attempt to destroy the aliens.” – Paul Erdős

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Table I. Known nontrivial values and bounds for two color
Ramsey numbers $R(k,l) = R(k,l;2)$.

**Definition**  The Ramsey numbers $R(k,k)$ are sometimes called the *diagonal Ramsey numbers*.

**Remarks**

- We can approximate an upper bound for $R(k,k)$ as follows:

$$R(k,k) \leq \binom{2k-2}{k-1} < \frac{2^{2k-2}}{|\text{all subsets of } \{1,...,2k-2\}|} < 2^{2k} = 4^k.$$

Diagonal Ramsey numbers grow at most exponentially.
• To prove a lower bound on a Ramsey number, we need to construct a specific red/blue edge-coloring of a complete graph such that there is no red $K_j$ or blue $K_k$. In general, this is difficult.

• Erdös’ remarkable insight for a lower bound on $R(k, k)$: construct a coloring at random and show that the probability that there is a monochromatic $K_k$ is less than 1. (This is an example of the probabilistic method.)