Ramsey theory (continued)

Recall that we ended last class discussing bounds on diagonal Ramsey numbers, \( R(k, k) \).

Remarks

- We can approximate an upper bound for \( R(k, k) \) as follows:

\[
R(k, k) \leq \binom{2k - 2}{k - 1} < \frac{2^{2k-2}}{\text{all subsets of } \{1, \ldots, 2k-2\}} < 2^k = 4^k.
\]

Diagonal Ramsey numbers grow at most exponentially.

- To prove a lower bound on a Ramsey number, we need to construct a specific red/blue edge-coloring of a complete graph such that there is no red \( K_j \) or blue \( K_k \). In general, this is difficult.

- Erdös’ remarkable insight for a lower bound on \( R(k, k) \): consider a coloring at random and show that the probability that there is a monochromatic \( K_k \) is less than 1. (This is an example of the probabilistic method.)

**Theorem** (Erdös). Let \( k \geq 2 \). Then

\[
R(k, k) \geq 2^{k/2} = \left(\sqrt{2}\right)^k.
\]

**Proof.** Note that \( R(2, 2) = 2 = 2^2/2 \), so the result holds for \( k = 2 \); we proceed with the proof assuming that \( k \geq 3 \).
Suppose \( n < 2^{k/2} \). Consider a random red/blue edge-coloring of \( K_n \); that is, with probability \( \frac{1}{2} \), color edge red, and with probability \( \frac{1}{2} \), color edge blue. (Flip a fair coin to determine color.)

Note that

\[
P[\exists \text{ red } K_k] \leq \frac{\binom{n}{k} \left( \frac{n}{2} - \binom{k}{2} \right)}{2^{(\frac{n}{2})}} = \binom{n}{k} 2^{-\binom{k}{2}}.
\]

Thus, we have

\[
P[\exists \text{ red } K_k \text{ or } \exists \text{ blue } K_k] \leq P[\exists \text{ red } K_k] + P[\exists \text{ red } K_k]
\leq 2 \left[ \binom{n}{k} 2^{-\binom{k}{2}} \right]
< 2 \frac{n^k}{k!} 2^{-\binom{k}{2}}
< 2 \frac{(2^{k/2})^k}{k!} 2^{-\binom{k}{2}} \quad \text{since } n < 2^{k/2}
= 2^{\frac{2^{k^2/2} - k(k-1)/2}{k!}}
= 2^{\frac{2^{k/2}}{k!}}.
\]

Let \( f(k) = 2^{\frac{2^{k/2}}{k!}} \). First, notice that

\[
\begin{align*}
f(3) &\approx 0.94, \\
f(4) &= \frac{1}{3}, \\
f(5) &\approx 0.094,
\end{align*}
\]
all of which are strictly less than 1.

Furthermore, since \( k! \geq \left( \frac{k}{2} \right)^{k/2} \), we have, for \( k \geq 6 \),

\[
f(k) = 2 \frac{2^{k/2}}{k!} \leq 2 \frac{2^{k/2}}{(\frac{k}{2})^{k/2}} = 2 \left( \frac{4}{k} \right)^{k/2} \leq 2 \left( \frac{2}{3} \right)^3 < 1.
\]

Thus, there is positive probability (> 0) that there exists neither a red \( K_k \) nor a blue \( K_k \). This implies there must exist a red/blue edge-coloring of \( K_n \) that contains neither a red \( K_k \) nor a blue \( K_k \) when \( n < 2^{k/2} \). Hence

\[
R(k, k) \geq 2^{k/2}.
\]

\[\blacksquare\]

**Remark**  We have

\[2^{k/2} < R(k, k) < 2^{2k}\]

and if we look at the logarithms of these values, we have

\[
\frac{k}{2} < \log_2 R(k, k) < 2k
\]

which yields

\[
\frac{1}{2} < \frac{\log_2 R(k, k)}{k} < 2.
\]

A major open problem in Ramsey theory: does the following limit exist, and if so, what is its value?

\[
\lim_{k \to \infty} \frac{\log_2 R(k, k)}{k}.
\]

We consider a natural generalization of the Ramsey numbers.
**Definition**  The *Ramsey number*, $R(p_1, p_2, \ldots, p_k)$, is the smallest integer $n$ such that every $k$-edge coloring of $K_n$ contains a complete subgraph of $p_i$ vertices whose edges are all colored with color $i$ for some $i$ where $1 \leq i \leq k$.

The special case we previously examined is that of $k = 2, p_1 = j$, and $p_2 = k$.

**Remarks**

- The only known value of a multicolor (three colors or more) classical Ramsey number is $R(3, 3, 3) = 17$.

The only two 3-edge-colorings of $K_{16}$ with no monochromatic $K_3$.

- Probably the most studied and intriguing open case is
  
  $51 \leq R(3, 3, 3, 3) \leq 62$.

- There is a generalization of a previous upper bound we proved last class (proof is a hmwk problem) : if $p_i \geq 2$ for all $i$,
  
  $R(p_1, \ldots, p_k) \leq -k + 2 + \sum_{i=1}^{k} R(p_1, \ldots, p_{i-1}, p_i - 1, p_{i+1}, \ldots, p_k)$.

We can generalize even further by looking for monochromatic subgraphs that need not be complete subgraphs.
**Definition** Given simple graphs $G_1, \ldots, G_k$, the *graph Ramsey number*, denoted $R(G_1, \ldots, G_k)$ is the smallest integer $n$ such that every $k$-edge coloring of $K_n$ contains a copy of $G_i$ in color $i$ for some $i$.

By “copy of $G_i$”, we mean a subgraph of $G$ isomorphic to $G_i$.

**Example** We claim that $R(P_3, K_3) = 5$.

Consider any red/blue edge-coloring of $K_5$. Let $v$ be any vertex.

**Case 1:** Vertex $v$ is incident to at least two red edges. Then $v$ with two neighbors along red edges form a red copy of $P_3$.

**Case 2:** Vertex $v$ is incident to at most 1 red edge, so $v$ is incident to at least 3 blue edges. Let $x, y, z$ be neighbors of $v$ along blue edges.

- If any pair amongst $x, y, z$ have a blue edge between them, then that pair of vertices along with $v$ form a blue $K_3$.
- Otherwise, $x, y, z$ have all red edges between them and hence form a red $P_3$.

Thus, $R(P_3, K_3) \leq 5$.

To see that $R(P_3, K_3) > 4$, consider the red/blue edge-coloring of $K_4$ shown below:
contains neither a red $P_3$ nor a blue $K_3$

Selected known results

- $R(P_n, P_m) = n + \left\lfloor \frac{m}{2} \right\rfloor - 1$ for $n \geq m \geq 2$.

- $R(K_{1,n}, K_{1,m}) = \begin{cases} m + n - 1 & \text{if } m, n \text{ both even} \\ m + n & \text{otherwise.} \end{cases}$

- $R(T_n, K_m) = (n - 1)(m - 1) + 1$, where $T_n$ is a tree on $n$ vertices.