Planar graphs

Wednesday, April 20, 2011
Math 55, Discrete Mathematics

Recall...
Theorem (Euler’s formula):
Let $G$ be a planar graph with $n$ vertices, $e$ edges, and $c$ components. Consider a planar embedding of $G$ with $f$ faces. Then
$$n - e + f - c = 1.$$ 

We prove the connected version of Euler’s formula... ...on the chalkboard!
More about faces

Implication of Euler’s formula:
Regardless of how you draw a planar graph without edge crossings, the number of faces is always the same.

degree of a face = the # of sides of edges on its boundary

Proposition: Let G be a planar graph. The sum of the degrees of the faces in a planar embedding of G is 2|E(G)|.

Bounds on number of edges in planar graph

Corollary (to Euler’s formula):
Let G be a planar graph with at least 2 edges. Then
|E(G)| ≤ 3|V(G)| - 6.

Furthermore, if G is triangle-free, then
|E(G)| ≤ 2|V(G)| - 4.

We prove the first part of this corollary...
...on the chalkboard!

Corollary (to Euler’s formula):
Let G be a planar graph. Then \( \delta(G) \leq 5 \).

Proof also on the chalkboard!
**Bound on min degree in a planar graph**

**Corollary (to Euler's formula):**
Let \( G \) be a planar graph. Then \( \delta(G) \leq 5 \).

**Characterizing planar graphs**

**Claim:** The graphs \( K_5 \) and \( K_{3,3} \) are nonplanar.

**Proof:** We have
\[
|E(K_5)| = 10 > 9 = 3|V(K_5)| - 6.
\]
Thus, \( K_5 \) cannot be a planar graph (by previous corollary).

Also, \( K_{3,3} \) is a bipartite graph, so it has no odd cycles, and thus, no triangles. Furthermore,
\[
|E(K_{3,3})| = 9 > 8 = 2|V(K_{3,3})| - 4.
\]
Thus, \( K_{3,3} \) cannot be a planar graph (by previous corollary).
Can you run connections from each utility plant to each home without any connections crossing?

Is $K_{3,3}$ planar?

Note: You may place the homes and utility plants anywhere you like (except on top of one another).
Characterizing planar graphs

**Theorem (Kuratowski’s Thm):**
A graph is planar if and only if it does not contain a subdivision of $K_5$ or $K_{3,3}$.

A subdivision of a graph $G$ is formed by replacing edges of $G$ with paths.

A subdivision of $K_5$
Example using subdivisions

A subdivision of a graph $G$ is formed by replacing edges of $G$ with paths.

Claim: graph has a subdivision of $K_{3,3}$
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Characterizing planar graphs

Theorem (Kuratowski’s Thm): A graph is planar if and only if it does not contain a subdivision of $K_5$ or $K_{3,3}$.

A subdivision of a graph $G$ is formed by replacing edges of $G$ with paths.

This theorem can also be used to show that the Petersen graph is a nonplanar graph.

In your spare time...

Planarity is a game based on planar graphs:

http://www.planarity.net/
**Planar graph proofs (done on chalkboard)**

**Theorem** (Euler’s formula). *If a connected planar graph $G$ with $n$ vtcs and $e$ edges has a planar embedding with $f$ faces, it follows that*

$$n - e + f = 2.$$

*Proof.* Proof by induction on $e = |E(G)|$. Fix the number of vtcs in $G$ as $n$.

**Base case:** Since $G$ is connected, it must have at least $n - 1$ edges, so base case is $e = n - 1$. Because $G$ is a connected graph with $n$ vtcs and $n - 1$ edges, it follows that $G$ is a tree. Since $G$ is acyclic, it has exactly one face in any planar embedding. So

$$n - e + f = n - (n - 1) + 1 = 2$$

as desired.

**Induction hypothesis:** Suppose the result holds for any connected, planar graph with $n$ vtcs and $e \geq n - 1$ edges.

Let $G$ be a connected planar graph with $n$ vtcs and $e + 1$ edges. Consider a planar embedding of $G$ with $f$ faces. Since $G$ is not a tree (it has at least $n$ edges), there exists an edge $uv$ of $G$ that is not a cut edge.

Consider graph $G - uv$. It is a connected graph with $n$ vtcs and $e$ edges. Furthermore, $G - uv$ is planar and the planar embedding of $G - uv$ (use the planar embedding of $G$ from above and “erase” edge $uv$) has $f - 1$ faces. This is a consequence of the fact that there are two distinct faces in the planar embedding of $G$ that have $uv$ on their boundaries. When $uv$ is removed, these two faces become one face.\(^1\)

Applying the induction hypothesis to $G - uv$, we have

$$n - e + (f - 1) = 2.$$

\(^1\)Formally, this requires some further justification, i.e. the Jordan Curve Theorem.
From rearranging terms in this expression, it follows that
\[ n - (e + 1) - f = 2, \]
which is the desired conclusion.

Thus, the result holds by induction. ■

We can use Euler’s formula to derive upper bounds on the number of edges in planar graphs (in terms of the number of vertices).

**Corollary.** Let \( G \) be a planar graph with \( n \) vertices and \( e \geq 2 \) edges. Then
\[ e \leq 3n - 6. \]
Furthermore, if \( G \) is triangle-free, then
\[ e \leq 2n - 4. \]

**Proof.** Without loss of generality, assume \( G \) is connected. (If not and \( G \) has \( c \) components, add \( c - 1 \) edges between the components to make it connected and maintain planarity.)

Consider a planar embedding of \( G \). Then, by Euler’s formula, we have
\[ n - e + f = 2 \implies f = 2 - n + e \quad (\ast). \]
Since \( G \) has more than one edge, every face in the embedding has degree at least 3. Thus,
\[ 2e = \text{sum of degrees of faces} \geq 3f = 3(2 - n + e) \quad \text{by (\ast)}. \]
Rearranging terms, we have
\[ 6 - 3n + 3e \leq 2e \quad \implies \quad e \leq 3n - 6. \]
The proof of the second part of the corollary is left as an exercise. (Based on the assumption that \( G \) is triangle-free, we adjust the bound on the degree of any face.) ■
Remark Notice that this necessary condition is not a sufficient condition for a graph to be planar. For example, consider the triangle-free graph $G$ below:

$$\text{graph } G$$

It has 7 vtc's and 10 edges. So $|E(G)| = 10 \leq 2 \cdot 7 - 4 = 2|V(G)| - 4$ but this graph contains $K_{3,3}$, which we already saw was nonplanar.

Corollary. Let $G$ be a planar graph. Then $\delta(G) \leq 5$.

Proof. Suppose $G$ is a planar graph with $n$ vertices. If $G$ has less than 2 edges, then clearly the result holds, so assume $G$ has at least 2 edges. Then

$$n\delta(G) \leq \sum_{v \in V(G)} d(v) = 2|E(G)| \leq 2(3n - 6) = 6n - 12.$$

We divide both sides by $n$ to obtain

$$\delta(G) \leq 6 - \frac{12}{n} < 6$$

where the last inequality follows from the fact that $n \geq 1 \Rightarrow \frac{12}{n} > 0$. Since $\delta(G)$ is an integer, we have $\delta(G) \leq 5$. 

An example for which $\delta(G) = 5$: icosahedral graph
Coloring planar graphs

Strategy for proof of 6-colorability: proof by induction; color min degree vertex with a leftover color.

Theorem (Six Color Theorem). Every planar graph is 6-colorable.
Proof. We prove by induction on $n = |V(G)|$.

Base cases ($n \leq 6$): Let $G$ be a planar graph with 6 vtc's or less. Then by assigning each vertex a distinct color, we have a proper 6-coloring of $G$.

Induction hypothesis: Any planar graph on $n$ vertices is 6-colorable.

Consider a planar graph $G$ on $n + 1$ vertices. By previous corollary, $G$ has a vertex $v$ such that $d(v) \leq 5$.

Let $G' = G - v$. Then $G'$ is planar and has $n$ vertices. By the IH, $G'$ is 6-colorable.

Properly color vtc's of $G'$ using 6 colors. Extend this coloring to $G$ by assigning $v$ a color that is not present among its neighbors. Note that this is possible since $v$ has at most 5 neighbors, and there are 6 colors to choose from, so at least one color is not present amongst neighbors of $v$. Hence, $G$ is 6-colorable.

Theorem (Five Color Theorem). Every planar graph is 5-colorable.

See slides.

Theorem (Four Color Theorem). Every planar graph is 4-colorable.
Coloring planar graphs

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Coloring planar graphs with 6 colors

**Theorem:** Every planar graph is 6-colorable.

**Proof type:** Induction on $n = |V(G)|$.

**Basic argument of inductive step:**
- Let $v$ be a vertex of min degree in a planar graph $G$ on $n + 1$ vtc.
- Consider $G-v$.
- Apply IH to $G-v$ to get a 6-coloring of $G-v$.
- Color min degree vertex $v$ with a leftover color.

**Key fact:**
- $\delta(G) \leq 5$. \\[4000]
Theorem: Every planar graph is 5-colorable.

Proof type: Induction on $n = |V(G)|$.

Basic argument: Color min degree vertex $v$ with a leftover color, if possible. Otherwise, swap two colors to make a color available at $v$.

Key facts:
- $\delta(G) \leq 5$.
- Cannot have both a 1-3 alternating path and a 2-4 alternating path.
Theorem: Every planar graph is 5-colorable.

Proof type: Induction on $n = |V(G)|$.

Basic argument: Color min degree vertex $v$ with a leftover color, if possible. Otherwise, swap two colors to make a color available at $v$.

Key facts:
• $\delta(G) \leq 5$.
• Cannot have both a 1-3 alternating path and a 2-4 alternating path.
Coloring planar graphs with 5 colors

**Theorem:** Every planar graph is 5-colorable.

**Proof type:** Induction on $n = |V(G)|$.

**Basic argument:** Color min degree vertex $v$ with a leftover color, if possible. Otherwise, swap two colors to make a color available at $v$.

**Key facts:**
- $\delta(G) \leq 5$.
- Cannot have both a 1-3 alternating path and a 2-4 alternating path.

Coloring planar graphs with 4 colors

**Theorem** (Four Color Thm): Every planar graph is 4-colorable.

**Proof:**
Proof by Appel and Haken in 1977 (computer-aided proof).

Note that it is easy to see that the above bound is best possible by considering $K_4$.

To show $\chi(G) \leq 4$ is **very** difficult.

A new (more simplified) proof of the Four Color Theorem was given in 1996 by Robertson, Sanders, Seymour, and Thomas.

There were several early failed attempts at proving the theorem.

**Timeline**

1879: Proof by Alfred Kempe, which was widely acclaimed.
1880: Proof by Peter Guthrie Tait.
1890: Kempe’s proof was shown incorrect by Percy Heawood. (Heawood proved 5-color theorem at this time.)
1891: Tait’s proof was shown incorrect by Julius Petersen.