Eulerian graphs

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Math 55, Discrete Mathematics

Bridges of Königsberg problem

Is there a walking route which crosses every bridge exactly once? Is there a walking route that begins and ends at the same place which crosses every bridge exactly once?
The Eulerian basics

- **trail**: a walk with no repeated edges
- **Eulerian trail**: a trail that includes all edges in $G$
- **Eulerian tour**: a closed Eulerian trail (begins and ends at same vertex)
- **Eulerian graph**: a graph that has an Eulerian tour

![Graph diagrams](image)

Necessary conditions

What are the necessary conditions for a graph $G$ to be Eulerian or to have an Eulerian trail?

- $G$ has only one nontrivial component (connected except for isolated vertices)
- $G$ has at most 2 vertices of odd degree to have an Eulerian trail
- $G$ has all vertices of even degree to have an Eulerian tour (even graph)
Necessary conditions

What are the necessary conditions for a graph $G$ to be Eulerian or to have an Eulerian trail?

- $G$ has only one nontrivial component (connected except for isolated vertices).
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- $G$ has all vertices of even degree to have an Eulerian tour (even graph).

The obvious necessary conditions are also sufficient!

Characterization of Eulerian tours and trails

**Theorem:** Let $G$ be a connected graph.

- If $G$ has all vertices of even degree, then there is an Eulerian tour that begins and ends at any vertex $v$.
- If $G$ has exactly two vertices, $a$ and $b$, of odd degree, then there is an Eulerian trail that begins at $a$ and ends at $b$. 
Bridges of Königsberg as a graph

Is there a walking route which crosses every bridge exactly once?

Does this multigraph have an Eulerian trail?

NO!

The multigraph has more than 2 vertices of odd degree.

Tools for proof of theorem

Lemma 1:
Let $G$ be a connected graph whose vertices all have even degrees. Then no edge of $G$ is a cut edge.

Lemma 2:
If $G$ is a connected graph with leaf $v$, then $G-v$ is connected.

Lemma 3:
Let $G$ be a connected graph with exactly 2 vertices, $a$ and $b$, of odd degree. Suppose $d(a) > 1$. Then at least one of the edges incident to $a$ is not a cut edge.
Proof by induction on # of edges.

**Base case:** $G$ has 0 edges. Then path of length 0, $v$, is Eulerian tour.

**One more base case:** $G$ has 1 edge and is connected. Then path $a, b$ is an Eulerian trail.

**Induction hypothesis (IH):** Assume $G$ is a connected graph with $m$ edges.

- If all vtc's have even degree, then $G$ has an Eulerian tour that begins and ends at any vertex $v$.
- If $G$ has exactly two vtc's, $a$ and $b$, of odd degree, then there is an Eulerian trail that begins at $a$ and ends at $b$.

Let $G$ be a connected graph with $m + 1$ edges.

**Case 1:** All vtc's have even degree.

Let $v$ be any vertex, and let $w$ be a neighbor of $v$ ($v \sim w$).

Consider $G' = G - vw$.
- $G'$ has $m$ edges,
- exactly 2 vtc's of odd degree ($v$ and $w$),
- and is connected (by Lemma 1).
Let $G$ be a connected graph with $m + 1$ edges.

**Case 1:** All vtcs have even degree.

Let $v$ be any vertex, and let $w$ be a neighbor of $v$ ($v \sim w$).

Consider $G' = G - vw$.

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- $G'$ has $m$ edges,
  - exactly 2 vtcs of odd degree ($v$ and $w$),
  - and is connected (by Lemma 1).

**Apply IH.**

-->$G - vw$ has Eulerian trail from $v$ to $w$.

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**Case 2:** $G$ has exactly two vtcs, $a$ and $b$, of odd degree.

**Case 2a:** $d(a) = 1$ and $d(b) = 1$.

If $a \sim b$, then $G$ is $K_2$ (base case). Done.

Let $x$ be unique neighbor of $a$ ($x \neq b$). Consider $G' = G - a$.

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- $G'$ has $m$ edges,
  - exactly 2 vtcs of odd degree ($x$ and $b$),
  - and is connected (by Lemma 2).

**Apply IH.**

-->$G' = G - a$ has Eulerian trail from $x$ to $b$.

Add edge $ax$ to start of trail to get Eulerian trail that begins at $a$ and ends at $b$. 

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Let $G$ be a connected graph with $m + 1$ edges.

**Case 2:** $G$ has exactly two vtcs, $a$ and $b$, of odd degree.

**Case 2b:** At least one of $a, b$ has degree $> 1$.

WLOG, assume $d(a) > 1$. Then $d(a) \geq 3$.

By Lemma 3, there is edge incident to $a$ that is not a cut edge; call it edge $ax$. Consider $G' = G - ax$.
Then $G'$ has $m$ edges and is connected.

If $x \neq b$, then $G'$ has exactly 2 vtcs of odd degree ($x$ and $b$).

Apply IH.

--- $G - ax$ has Eulerian trail from $x$ to $b$.
Add edge $ax$ to start of trail to get Eulerian trail that begins at $a$ and ends at $b$.

If $x = b$, then $G'$ has all vtcs of even degree.

Apply IH.

--- $G - ab$ has Eulerian tour that begins and ends at $b$.
Add edge $ax (= ab)$ to start of tour to get Eulerian trail that begins at $a$ and ends at $b$.

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**Proof of Lemma 1**

**Lemma 1:**
Let $G$ be a connected graph whose vtcs all have even degrees. Then no edge of $G$ is a cut edge.

**Proof:** Let $G$ be a connected even graph.

Suppose BWOC that $e = xy$ is a cut edge of $G$.
Then $G - e$ has exactly 2 components (by previous thm).
Each component has exactly 1 vertex of odd degree.

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(Hmwk problem: Any graph has an even number of vtcs of odd degree.)
Proof of Lemma 2

**Lemma 2:**
If $G$ is a connected graph with leaf $v$, then $G - v$ is connected.

**Proof:**
Let $G$ be a connected graph with leaf $v$, and let $a, b \in V(G - v)$. We want to exhibit a path from $a$ to $b$ in $G - v$.

$G$ connected --> there is a path $P$ from $a$ to $b$ in $G$.

If $P$ contains $v$, then $P$ looks like $a, ..., v, ... b$ and since $v \neq a$ and $v \neq b$, $v$ has 2 distinct neighbors, contradicting fact that $d(v) = 1$ in $G$.

Thus, $P$ is an $a, b$-path in $G - v$, and so $G - v$ is connected.

Proof of Lemma 3

**Lemma 3:**
Let $G$ be a connected graph with exactly 2 vtcs, $a$ and $b$, of odd degree. Suppose $d(a) > 1$. Then at least one of the edges incident to $a$ is not a cut edge.

**Proof:**
Suppose, BWOC, that all edges incident to $a$ are cut edges.

$G$ connected --> there is a path $P$ from $a$ to $b$ in $G$;

let $e$ be an edge incident to $a$ that is not used in $P$.

Consider graph $G' = G - e$. Then $G'$ has 2 components (by previous thm). Since path $P$ still exists in $G'$, $a$ and $b$ are in same component of $G'$.

Vertex $a$ has even degree in $G'$, and all other vtcs have same degrees in $G'$ as in $G$.  --><--
(component containing $a$ in $G'$ has one vertex of odd degree)
Fleury’s algorithm to find an Eulerian tour

Don’t make any blatant mistakes!

Fleury’s algorithm to find an Eulerian tour

1. Pick any vertex as a starting point.

2. Marking your path as you move from vertex to vertex, travel along any edges you wish with one exception:

   do not travel along an edge that is a cut edge for the graph formed by the not-yet-traversed edges (unless you have no other choice).

3. Continue until you return to your starting point.
Fleury’s algorithm in action

start at f

traverse edge fc

traverse edge cd

Fleury’s algorithm in action

traverse edge da
Fleury’s algorithm in action

Do not traverse edge ab since this is a cut edge of not-yet-traversed graph!

Fleury’s algorithm in action

traverse edge da

traverse edge ac
Fleury’s algorithm in action

traverse edge da

traverse edge ac

Rest of tour is clear since there is only one choice at each step.

Fleury’s algorithm in action

Eulerian tour: f, c, d, a, c,
Fleury’s algorithm in action

traverse edge df

Eulerian tour: f, c, d, a, c, e, a, b, d, f

* To try Fleury’s algorithm yourself, check out http://www.cut-the-knot.org/Curriculum/Combinatorics/FleuryAlgorithm.shtml