1.1.14 Prove that removing opposite corner squares from an 8-by-8 checkerboard leaves a subboard that cannot be partitioned into 1-by-2 and 2-by-1 rectangles. Using the same argument, make a general statement about all bipartite graphs.

1.1.15 Consider the following four families of graphs: A = \{ paths \}, B = \{ cycles \}, C = \{ complete graphs \}, D = \{ bipartite graphs \}. For each pair of these families, determine all isomorphism classes of graphs (i.e., all unlabeled graphs) that belong to both families.

1.1.23(c) Among simple graphs, determine the smallest $n$ such that there exist nonisomorphic $n$-vertex graphs having the same list of vertex degrees.

1.1.31 A graph $G$ is self-complementary if it is isomorphic to its complement. Examples of self-complementary graphs include $P_4$ and $C_5$.

Prove that a self-complementary graph with $n$ vertices exists if and only if $n$ or $n - 1$ is divisible by 4. (Hint: When $n$ is divisible by 4, generalize the structure of $P_4$ by splitting the vertices into four groups. For $n \equiv 1 \mod 4$, add one vertex to the graph constructed for $n - 1$.)

1.2.22 Prove that a graph $G$ is connected if and only if for every partition of its vertices into two nonempty sets, there is an edge with endpoints in both sets.

1.3.17 Let $G$ be a graph with at least two vertices. Prove or disprove:

(a) Deleting a vertex of degree $\Delta(G)$ cannot increase the average degree.

(b) Deleting a vertex of degree $\delta(G)$ cannot reduce the average degree.
Information you may find helpful:

Special classes of graphs:

- A path graph is a graph whose vertices can be labeled \( v_1, v_2, \ldots, v_n \) such that \( v_i \sim v_{i+1} \) for \( 1 \leq i \leq n - 1 \) and no other edges exist in the graph. A path graph on \( n \) vertices is denoted \( P_n \).

  Figure 1: The path graphs \( P_1, P_2, P_3, P_4, \) and \( P_5 \).

- A cycle graph is a graph with at least 3 vertices whose vertices can be labeled \( v_1, v_2, \ldots, v_n \) such that \( v_i \sim v_{i+1} \) for \( 1 \leq i \leq n - 1 \), \( v_1 \sim v_n \), and no other edges exist in the graph. A cycle graph on \( n \) vertices is denoted \( C_n \).

  Figure 2: The cycle graphs \( C_3, C_4, \) and \( C_5 \).

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\(^1\)Since we are in the context of simple graphs, the smallest cycle is one with 3 vertices. If, on the other hand, we allowed loops and multiple edges, we could have a cycle graph with one vertex or two vertices.