Some Important Results from Analysis
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Definitions: Let \( \{a_n\} \) be a sequence of real numbers. Then we let

\[
\liminf_{n \to \infty} a_n = \lim_{n \to \infty} \left( \inf_{\nu \geq n} a_\nu \right) = \sup_{n} \left( \inf_{\nu \geq n} a_\nu \right).
\]

Note that if \( \{a_n\} \) is monotone, then

\[
\liminf_{n \to \infty} a_n = \lim_{n \to \infty} a_n.
\]

Similarly, we let

\[
\limsup_{n \to \infty} a_n = \lim_{n \to \infty} \left( \sup_{\nu \geq n} a_\nu \right) = \inf_{n} \left( \sup_{\nu \geq n} a_\nu \right).
\]

Once again we see that if \( \{a_n\} \) is monotone, then

\[
\limsup_{n \to \infty} a_n = \lim_{n \to \infty} a_n.
\]

Observe that for every \( n \),

\[
\inf_{\nu \geq n} a_\nu \leq a_n \leq \sup_{\nu \geq n} a_\nu
\]

and hence

\[
\liminf_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_n.
\]

Finally, observe that

\[
\liminf_{n \to \infty} (-a_n) = -\limsup_{n \to \infty} a_n.
\]

Main Theorem: If \( \{a_n\} \) is a sequence of real numbers, then

\[
L = \lim_{n \to \infty} a_n \text{ exists, possibly } +\infty \text{ or } -\infty,
\]

if and only if \( \liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n = L \).

Proof: If \( \lim_{n \to \infty} a_n = +\infty \) then, for every real number \( M \),

\[
n \geq N \implies a_n \geq M.
\]
Therefore, 
\[ n \geq N \implies (\inf_{\nu \geq n} a_{\nu}) \geq M. \]

Consequently, 
\[ \liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n = +\infty. \]

Conversely, if \( \liminf_{n \to \infty} a_n = +\infty \) then, for every real number \( M \), there exists a positive integer \( N \) such that 
\[ n \geq N \implies (\inf_{\nu \geq n} a_{\nu}) \geq M. \]

Hence, 
\[ n \geq N \implies a_n \geq M, \]
and thus \( \lim_{n \to \infty} a_n = +\infty \).

A similar argument works for \( L = -\infty \), so now assume \( -\infty < L < +\infty \). If \( \lim_{n \to \infty} a_n = L \), then for every \( \epsilon > 0 \) there exists a positive integer \( N \) such that 
\[ n \geq N \implies L - \epsilon \leq a_n \leq L + \epsilon. \]

Therefore, 
\[ n \geq N \implies L - \epsilon \leq (\inf_{\nu \geq n} a_{\nu}) \leq (\sup_{\nu \geq n} a_{\nu}) \leq L + \epsilon, \]
and, hence, 
\[ \liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n = L. \]

Conversely, if this relation is true, then for every \( \epsilon > 0 \) there exists a positive integer \( N \) such that 
\[ n \geq N \implies L - \epsilon \leq (\inf_{\nu \geq n} a_{\nu}) \leq (\sup_{\nu \geq n} a_{\nu}) \leq L + \epsilon \]
and this shows that 
\[ n \geq N \implies L - \epsilon \leq a_n \leq L + \epsilon. \]

Consequently, \( \lim_{n \to \infty} a_n = L \).

**Fatou's Lemma for Series:** Let \( \{a_{n,k}\} \) be a double sequence of non-negative real numbers with \( \liminf_{n \to \infty} a_{n,k} = b_k \) for every \( k \). Then,
\[ \sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \liminf_{n \to \infty} a_{n,k} \leq \liminf_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k}. \]

**Proof:** For every \( k \) and \( \mu \geq n \),
\[ \inf_{\nu \geq n} a_{\nu,k} \leq a_{\mu,k}. \]
Consequently, for every \( m \) and \( \mu \geq n \),
\[
\sum_{k=1}^{m} \inf_{\nu \geq n} a_{\nu,k} \leq \sum_{k=1}^{m} a_{\mu,k},
\]
which means that for every \(m\) and \(n\),
\[
\sum_{k=1}^{m} \inf_{\nu \geq n} a_{\nu,k} \leq \inf_{\mu \geq n} \sum_{k=1}^{m} a_{\mu,k}.
\]
Therefore, for every \(m\) we have,
\[
\sum_{k=1}^{m} b_k = \sum_{k=1}^{m} \lim_{n \to \infty} \left( \inf_{\nu \geq n} a_{\nu,k} \right) = \lim_{n \to \infty} \sum_{k=1}^{m} \inf_{\nu \geq n} a_{\nu,k} \leq \lim_{n \to \infty} \left( \inf_{\mu \geq n} \sum_{k=1}^{\infty} a_{\mu,k} \right) = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k}.
\]

**Monotone Convergence Theorem for Series:** Let \(\{a_{n,k}\}\) be a double sequence of non-negative real numbers such that for every \(k\) we have \(a_{n,k} \uparrow b_k\) as \(n \to \infty\). Then
\[
\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \lim_{n \to \infty} a_{n,k} = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k}.
\]

**Proof:** From Fatou's Lemma for Series we know that
\[
\sum_{k=1}^{\infty} b_k \leq \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k},
\]
since \(\{\sum_{k=1}^{\infty} a_{n,k}\}\) is monotone increasing in \(n\). On the other hand, we have for every \(n\) and \(k\),
\[
a_{n,k} \leq b_k\text{ which implies that } \sum_{k=1}^{\infty} a_{n,k} \leq \sum_{k=1}^{\infty} b_k \text{ for every } n.
\]

An important application is:

**Monotone Convergence Theorem for non-negative integer valued random variables:** Let \(\{Y_n\}\) be a sequence of non-negative integer valued random variables and suppose that \(Y_n \uparrow Y\) with probability 1 as \(n \to \infty\). Then
\[
E(Y) = E(\lim_{n \to \infty} Y_n) = \lim_{n \to \infty} E(Y_n).
\]

**Proof:** Let \(a_{n,k} = P(Y_n \geq k)\). Since \(\{Y_n \geq k\} \uparrow \{Y \geq k\}\) as \(n \to \infty\) for every \(k\), \(a_{n,k} \uparrow P(Y \geq k)\). Thus
\[
E(Y) = \sum_{k=1}^{\infty} P(Y \geq k) = \lim_{n \to \infty} \sum_{k=1}^{\infty} P(Y_n \geq k) = \lim_{n \to \infty} E(Y_n).
\]
Corollary: If \( \{N_j\} \) is a sequence of non-negative integer valued random variables and we define \( Y_n = \sum_{j=1}^{n} N_j, \ Y = \sum_{j=1}^{\infty} N_j \), then
\[
E(\sum_{j=1}^{\infty} N_j) = \sum_{j=1}^{\infty} E(N_j).
\]

Proof:
\[
E(\sum_{j=1}^{\infty} N_j) = E(Y) = \lim_{n \to \infty} E(Y_n) = \lim_{n \to \infty} \sum_{j=1}^{n} E(N_j) = \sum_{j=1}^{\infty} E(N_j).
\]

Dominated Convergence Theorem for Series: Let \( \{a_{n,k}\} \) be a double sequence of real numbers with
\[
\lim_{n \to \infty} a_{n,k} = b_k \text{ for every } k.
\]
Suppose further that for every \( n \) and \( k \),
\[
|a_{n,k}| \leq d_k \text{ where } \sum_{k=1}^{\infty} d_k < \infty,
\]
that is, all of the sequences \( \{a_{n,k}\}_{k=1}^{\infty} \) for \( n = 1, 2, 3, \ldots \) are dominated by the non-negative sequence \( \{d_k\} \) which has a finite sum. Then
\[
\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \lim_{n \to \infty} a_{n,k} = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k}.
\]

Proof: First note that for \( n = 1, 2, 3, \ldots \), the series \( \sum_{k=1}^{\infty} a_{n,k} \) are absolutely convergent and hence convergent, since \( \sum_{k=1}^{\infty} d_k < \infty \). Furthermore, \( \lim_{n \to \infty} a_{n,k} = b_k \) for every \( k \) means that \( |b_k| \leq d_k \) for every \( k \) and thus the series \( \sum_{k=1}^{\infty} b_k \) is also absolutely convergent and therefore convergent. Finally, observe that since
\[
-d_k \leq a_{n,k} \leq d_k
\]
for every \( n \) and \( k \), the sequences \( \{d_k - a_{n,k}\}_{k=1}^{\infty} \) and \( \{d_k + a_{n,k}\}_{k=1}^{\infty} \) are non-negative with
\[
\lim_{n \to \infty} (d_k - a_{n,k}) = d_k - b_k \text{ and } \lim_{n \to \infty} (d_k + a_{n,k}) = d_k + b_k
\]
for every \( k \). Hence, by Fatou’s Lemma for Series we have
\[
\sum_{k=1}^{\infty} (d_k - b_k) \leq \lim \inf_{n \to \infty} \sum_{k=1}^{\infty} (d_k - a_{n,k}) \text{ and } \sum_{k=1}^{\infty} (d_k + b_k) \leq \lim \inf_{n \to \infty} \sum_{k=1}^{\infty} (d_k + a_{n,k}).
\]
In other words,
\[
\sum_{k=1}^{\infty} d_k - \sum_{k=1}^{\infty} b_k \leq \sum_{k=1}^{\infty} d_k + \lim \inf_{n \to \infty} (- \sum_{k=1}^{\infty} a_{n,k})
\]
and

\[ \sum_{k=1}^{\infty} d_k + \sum_{k=1}^{\infty} b_k \leq \sum_{k=1}^{\infty} d_k + \liminf_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k}. \]

Therefore,

\[ \sum_{k=1}^{\infty} b_k \leq \liminf_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k} \]

\[ \leq \limsup_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k} \]

\[ = - \liminf_{n \to \infty} \left( - \sum_{k=1}^{\infty} a_{n,k} \right) \]

\[ \leq \sum_{k=1}^{\infty} b_k. \]