Construction of Brownian Motion

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Fundamental Lemma: Suppose that $s < t$, and $X(s)$ and $X(t)$ are random variables defined on the same sample space such that $X(t) - X(s)$ has a distribution which is $N(0, t-s)$. Then there exists a random variable $X(\frac{t+s}{2})$ such that $X(\frac{t+s}{2}) - X(s)$ and $X(t) - X(\frac{t+s}{2})$ are independent with a common $N(0, \frac{t-s}{2})$ distribution.

Proof: Let $U = X(t) - X(s)$. Suppose V is independent of U and also has a $N(0, t-s)$ distribution. Define $X(\frac{t+s}{2})$ by the equations:

$$\begin{aligned} X(t) - X(\frac{t+s}{2}) &= \frac{U+V}{2}, \\ X(\frac{t+s}{2}) - X(s) &= \frac{U-V}{2}. \end{aligned}$$

Thus in addition to $U = X(t) - X(s)$ we also have

$$V = X(t) + X(s) - 2X(\frac{t+s}{2})$$

or, equivalently,

$$\frac{X(t) + X(s)}{2} - X(\frac{t+s}{2}) = \frac{1}{2}V.$$

Consequently, $X(\frac{t+s}{2}) - X(s)$ and $X(t) - X(\frac{t+s}{2})$ have a common $N(0, \frac{t-s}{2})$ distribution. Furthermore, they are independent, since $U+V$ and $U-V$ are uncorrelated. To see this, note that

$$E\left((U+V)(U-V)\right) = E(U^2) - E(V^2) = 0.$$

Construction of Standard Brownian Motion on [0,1]: Here's the basic idea. For every non-negative integer n we will construct a continuous path Gaussian process

$$\left\{B^{(n)}(t) : 0 \le t \le 1\right\}$$

which agrees with standard Brownian motion on the binary rational points $\left\{\frac{k}{2^n}\right\}$ in $[0,1]$ and has the property that its value doesn't change at these points when n is increased to $n+1$. That is,

$$B^{(n+1)}(\frac{k}{2^n}) = B^{(n+1)}(\frac{2k}{2^{n+1}}) = B^{(n)}(\frac{k}{2^n})$$

for $k = 0, 1, \ldots, 2^n$ and $n = 0, 1, 2, \ldots$.

We begin with a sequence of independent random variables

$$\left\{ V(1),\ V(\frac{2k+1}{2^{n+1}}) : k = 0, 1, \ldots, (2^n - 1),\ n = 0, 1, 2, \ldots \right\}$$

where $V(1)$ has a $N(0,1)$ distribution and each $V(\frac{2k+1}{2^{n+1}})$ has a $N(0, \frac{1}{2^n})$ distribution. Thus, as examples, for $n = 0$ we have $V(\frac{1}{2})$ with a $N(0,1)$ distribution, and for $n = 1$ we have $V(\frac{1}{4})$ and $V(\frac{3}{4})$ with $N(0, \frac{1}{2})$ distributions.

The construction is done by strong induction. To begin the process we define $X(0) = 0$ and $X(1) = V(1)$. We next use the above lemma to construct $X(\frac{1}{2})$ from $X(1) - X(0) = V(1)$ and $V(\frac{1}{2})$ so that $X(\frac{1}{2}) - X(0)$ and $X(1) - X(\frac{1}{2})$ are independent and each have a $N(0, \frac{1}{2})$ distribution.

We now assume the strong induction hypothesis, namely: Suppose that for some $n \geq 0$,

$$\left\{ X(\frac{k}{2^n}) : k = 0, 1, \ldots, 2^n \right\}$$

has been defined using

$$\left\{ V(\frac{k}{2^n}) : k = 0, 1, \ldots, 2^n \right\}$$

in such a way that

$$\left\{ X(\frac{k}{2^n}) - X(\frac{k-1}{2^n}) : k = 1, 2, \ldots, 2^n \right\}$$

are i.i.d. $N(0, \frac{1}{2^n})$ random variables. Note that we have done this for $n = 0$ and $n = 1$. Further, note that for $j = 2k$,

$$X(\frac{j}{2^{n+1}}) = X(\frac{2k}{2^{n+1}}) = X(\frac{k}{2^n})$$

has already been defined for $k = 0, 1, \ldots, 2^n$. Hence, for each $k = 0, 1, \ldots, 2^n - 1$ we construct $X(\frac{2k+1}{2^{n+1}})$ using $V(\frac{2k+1}{2^{n+1}})$ so that

$$\begin{aligned} X(\frac{2k+1}{2^{n+1}}) - X(\frac{2k}{2^{n+1}}) &= X(\frac{2k+1}{2^{n+1}}) - X(\frac{k}{2^n}) \ \text{ and} \\ X(\frac{2k+2}{2^{n+1}}) - X(\frac{2k+1}{2^{n+1}}) &= X(\frac{k+1}{2^n}) - X(\frac{2k+1}{2^{n+1}}) \end{aligned}$$

are $N(0, \frac{1}{2^{n+1}})$ random variables with

$$\left\{ X(\frac{k}{2^{n+1}}) - X(\frac{k-1}{2^{n+1}}) : k = 1, 2, \ldots, 2^{n+1} \right\}$$

independent.

Next, for $n = 0, 1, 2, \ldots$, we define the process $\left\{B^{(n)}(t) : 0 \le t \le 1\right\}$ by letting $B^{(n)}(t) = X(t)$ for $t \in \left\{\frac{k}{2^n} : k = 0, 1, \ldots, 2^n\right\}$ and making $B^{(n)}(t)$ linear on each interval $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ for $k = 0, 1, \ldots, 2^n - 1$. Thus each process $\{B^{(n)}(t) : 0 \le t \le 1\}$ is Gaussian (since for t in the interval $[\frac{k}{2^n}, \frac{k+1}{2^n}]$,

$$B^{(n)}(t) = X(\frac{k}{2^n}) + 2^n(t - \frac{k}{2^n})[X(\frac{k+1}{2^n}) - X(\frac{k}{2^n})]$$

and hence is a linear combination of independent normal random variables). Furthermore, this process has continuous sample paths. Define $\Delta^{(n)}(t) = B^{(n+1)}(t) - B^{(n)}(t)$ for $0 \le t \le 1$. Then let

$$\delta^{(n)} = \max_{0 \le t \le 1} |\Delta^{(n)}(t)| = \max_{0 \le k < 2^n} \max_{t \in [\frac{k}{2^n}, \frac{k+1}{2^n}]} |\Delta^{(n)}(t)|.$$

However, we see that

$$\begin{aligned} \max_{t \in [\frac{k}{2^n}, \frac{k+1}{2^n}]} |\Delta^{(n)}(t)| &= \left| \frac{1}{2}\left(X(\frac{k+1}{2^n}) + X(\frac{k}{2^n}) \right) - X(\frac{2k+1}{2^{n+1}}) \right| \\ &= \frac{1}{2}\left| V(\frac{2k+1}{2^{n+1}}) \right|, \end{aligned}$$

so that

$$\delta^{(n)} = \frac{1}{2} \max_{0 \le k < 2^n} \left| V(\frac{2k+1}{2^{n+1}}) \right|.$$

In particular, this shows that $\delta^{(n)}$ is a random variable with distribution the same as 1/2 the maximum of the absolute value of 2^n independent $N(0, \frac{1}{2^n})$ random variables.

We now show that with probability one, $\{B^{(n)}(t)\}$ converges *uniformly* on $[0, 1]$. The method is to show that with probability one, $\left\{B^{(n)}(t)\right\}$ is a Cauchy sequence with respect to uniform convergence on $[0, 1]$. To see this, we first estimate, for $x > 0$,

$$\begin{aligned} P\left(\delta^{(n)} > \frac{x/2}{\sqrt{2^n}}\right) &= P\left(\max_{0 \le k < 2^n} \left| V(\frac{2k+1}{2^{n+1}}) \right| /2 > \frac{x/2}{\sqrt{2^n}} \right) \\ &= P\left(\bigcup_{k=0}^{2^n - 1} \left\{ \left| V(\frac{2k+1}{2^{n+1}}) \right| > \frac{x}{\sqrt{2^n}} \right\} \right) \\ &\le \sum_{k=0}^{2^n - 1} P\left(\left| V(\frac{2k+1}{2^{n+1}}) \right| > \frac{x}{\sqrt{2^n}} \right) \\ &= 2^n P(|Z| > x) \\ &= 2^{n+1} P(Z > x), \end{aligned}$$

where Z has a $N(0, 1)$ distribution. Consequently, if for $n = 1, 2, 3, \ldots$ we let $x_n = 2\sqrt{n}$ and use the estimate

$$P(Z > x_n) = 1 - \Phi(x_n) \le \frac{1}{x_n}\varphi(x_n) = \frac{1}{x_n\sqrt{2\pi}} e^{-x_n^2/2},$$

we see that

$$P\left(\delta^{(n)} > \frac{\sqrt{n}}{\sqrt{2^n}}\right) \leq 2^{n+1}\frac{1}{2\sqrt{2n\pi}}e^{-2n} \leq \frac{1}{\sqrt{2\pi}}(2/e^2)^n.$$

Therefore,

$$\sum_{n=1}^{\infty} P\left(\delta^{(n)} > \frac{\sqrt{n}}{\sqrt{2^n}}\right) \leq \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}(2/e^2)^n < \infty$$

and, hence,

$$P\left(\{\delta^{(n)} > \frac{\sqrt{n}}{\sqrt{2^n}}\}\ i.o.\right) = 0.$$

Consequently, with probability one, there is a N such that for all $n > N$ we must have $\delta^{(n)} \leq \frac{\sqrt{n}}{\sqrt{2^n}}$ and thus, again with probability one,

$$\sum_{n=1}^{\infty} \delta^{(n)} < \infty.$$

However, if $n < m$, then, with probability one,

$$\max_{0\leq t\leq 1} |B^{(m+1)}(t) - B^{(n)}(t)| \leq \sum_{m=n}^{\infty} \delta^{(m)} \to 0 \text{ as } n \to \infty.$$

Since the continuous functions on $[0, 1]$ form a complete metric space with respect to uniform convergence, $\lim_{n\to\infty} B^{(n)}(t)$ exists uniformly on $[0, 1]$ and represents a continuous function on $[0, 1]$ with probability one. Thus we can define $\{B(t) : 0 \leq t \leq 1\}$ by

$$B(t) = \begin{cases} \lim_{n\to\infty} B^{(n)}(t), & \text{when this limit exists uniformly on } [0,1] \\ 0, & \text{otherwise (on a set of probability zero)} \end{cases}.$$