1. **January 19, 2005** A Mathematical Model for Probability

(a) **Reading**: Chapter 1 of the text and the handout “Review of Definitions for Probability”.

(b) **Text Exercises**: 1.6, 1.8, 1.9, 1.12, 1.15.

(c) **Problem to hand in**: Due on Friday, January 21, 2005. Suppose that \( P \) is finitely additive on an algebra (field) \( A \). Show that \( P \) is countably additive on \( A \) \iff whenever \( \{A_n\} \) is a sequence in \( A \) and \( A_n \downarrow \emptyset \), then \( P(A_n) \downarrow 0 \).

2. **January 21, 2005** Conditional Probability and Bayes’ Theorem

(a) **Reading**: Sections 2.1 and 2.2 in Chapter 2 of the text and the handout “Important Results from Analysis”.

(b) **Text Exercises**: 2.2, 2.5, 2.6 (known in the U.S.A. as the Monty Hall problem), 2.8, 2.12.

(c) **Problem to hand in**: Due on Monday, January 24, 2005. Exercise 2.11.

3. **January 24, 2005** Independence and the Borel-Cantelli Lemmas

(a) **Reading**: Sections 2.3 and 2.4 in Chapter 2 of the text and the handout “The Borel-Cantelli Lemmas”.

(b) **Text Exercises**: 2.13, 2.15, 2.17, 2.18.

(c) **Problems to hand in**: Due on Friday, January 28, 2005.

i. Your opponent specifies 3 successives results of tosses of a coin, e.g. HHT. You then specify another such result, e.g. THT. The winner is the person whose sequence appears first when a fair coin is tossed successively and independently. Find the strategy which will allow you, the second player, to win at least 2/3 of the time. More specifically, for each possible choice of your opponent, find the choice that gives you the maximum probability of winning and calculate that probability.
ii. Consider an infinite sequence of independent tosses of a fair coin. Let \( \{k_n\} \) be a sequence of positive integers with \( \lim_{n \to \infty} k_n = \infty \), e.g. \( k_n = n \). Let \( A_1 \) be the event that no H’s occur in the first \( k_1 \) tosses. Let \( A_2 \) be the event that no H’s occur in the next \( k_2 \) tosses, etc. Show that the \( \{A_n\} \) are mutually independent. Also, show that \( \lim_{n \to \infty} P(A_n) = 0 \) so that \( P(\liminf_{n \to \infty} A_n) = 0 \), i.e. the probability that all but a finite number of the \( A_n \) occur is 0. However, give examples of sequences \( \{k_n\} \) such that:

A. \( P(\limsup_{n \to \infty} A_n) = 0 \), i.e. the probability that an infinite number of the \( A_n \) occur is 0.

B. \( P(\limsup_{n \to \infty} A_n) = 1 \), i.e. the probability that an infinite number of the \( A_n \) occur is 1.

Hint: Use the Borel-Cantelli lemmas.

4. January 26, 2005 Discrete Distributions and Random Variables

(a) \textbf{Reading:} Sections 3.1 and 3.2 in Chapter 3 and Sections 4.1 and 4.2 in Chapter 4 of the text. Use Sections 9.1 and 9.3A,B in the appendix for reference.

(b) \textbf{Text Exercises:} 3.1, 3.7, 3.8, 4.2, 4.4.

(c) \textbf{Problem to hand in:} Due on Friday, January 28, 2005. Exercise 3.6. Bonus: show that this distance is actually a metric on the space of all probability measures on the same finite or countably infinite set.

5. January 28, 2005 Expected Values and Generating Functions

(a) \textbf{Reading:} The first two theorems of the handout “More Important Results from Analysis” and the first three pages of the handout “Probability Generating Functions”.

(b) \textbf{Text Exercises:} 3.3, 3.5, 4.1, 4.5, 4.8.

(c) \textbf{Problem to hand in:} Due on Monday, January 31, 2005. Exercise 4.9. The result of this exercise is that if \( h(x) = \exp(e^x - 1) \), then \( B_n = h^{(n)}(0) \) for \( n = 0, 1, 2, \ldots \). Give an independent proof of this by showing (by induction) that

\[
    h^{(n)}(x) = h(x) \sum_{k=0}^{n} S(n, k) e^{kx}
\]

where the recurrence

\[
    S(n + 1, k) = kS(n, k) + S(n, k - 1)
\]

holds for all integers \( n \geq 0 \) and \( k \geq 1 \) (provided we define \( S(0, 0) = 1 \), \( S(n, k) = 0 \) for \( k > n \), and, for \( n \geq 1 \), let \( S(n, 0) = 0 \)).

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