We are given the joint probability generating function of the non-negative integer valued random variables \( N \) and \( M \), namely:

\[
g_{N,M}(s, t) = E(s^N t^M) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} s^j t^k P(N = j, M = k).
\]

Now, if \( N \) and \( M \) are independent, then for all non-negative integers \( j \) and \( k \), we have

\[
P(N = j, M = k) = P(N = j)P(M = k),
\]

so that for all \( s \) and \( t \),

\[
g_{N,M}(s, t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} s^j t^k P(N = j)P(M = k) = \sum_{j=0}^{\infty} s^j P(N = j) \sum_{k=0}^{\infty} t^k P(N = k) = g_N(s)g_M(t)
\]

For the converse, we want to show that if

\[
g_{N,M}(s, t) = g_N(s)g_M(t)
\]

for all \( s \) and \( t \), then \( N \) and \( M \) are independent. To do this we first need to figure out how to compute the coefficients in a double power series \( g(s, t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} s^j t^k p_{j,k} \) in terms of the partial derivatives of the sum function \( g \) evaluated at \((0, 0)\). In other words, we need to show that the coefficients in a double power series are uniquely determined by the sum function, just as in the case of a single variable series. To do this, let \( n \) and \( m \) be non-negative integers and consider

\[
\frac{\partial^{n+m}}{\partial s^n t^m} s^j t^k p_{j,k}.
\]

Note that this derivative is identically 0 unless both \( j \geq n \) and \( k \geq m \). If these inequalities are satisfied, then

\[
\frac{\partial^{n+m}}{\partial s^n t^m} s^j t^k p_{j,k} = \frac{j! k!}{(j-n)!(k-m)!} s^{j-n} t^{k-m} p_{j,k}.
\]

So, when this derivative is evaluated at \((0, 0)\) the result is 0 unless \( j = n \) and \( k = m \) in which case the result is \( n!m!p_{n,m} \).
To summarize, if we differentiate $g(s,t)$ $n$ times with respect to $t$, $m$ times with respect to $s$, and then evaluate at $(0,0)$, we get $n!m!$ times the coefficient $p_{n,m}$ of $s^n t^m$ in the double series expansion of $g$. Consequently,

$$\frac{\partial^{n+m}}{\partial s^n t^m} g_{N,M}(s,t)|_{(0,0)} = n!m! P(N = n, M = m).$$

If we now assume that $g_{N,M}(s,t) = g_N(s) g_M(t)$, then

$$\frac{\partial^{n+m}}{\partial s^n t^m} g_{N,M}(s,t)|_{(0,0)} = g_N^{(n)}(0) g_M^{(m)}(0) = n! P(N = n) m! P(M = m),$$

using what we know about individual probability generating functions (i.e. Taylor’s theorem for functions of one variable). Therefore, the relation $P(N = n, M = m) = P(N = n) P(M = m)$ holds for all non-negative integers $n$ and $m$, which means that $N$ and $M$ are independent.