Suppose that, as in Example 6.21, there is a safe haven with a compound growth rate of \(\theta > 0\), while the risky investment has probability \(p\) of turning unit amount into \(1 + u\) with \(u > \theta\), and probability \(q = 1 - p\) of turning unit amount into \(1 - d\) with \(0 < d \leq 1\). In each year, you place a fraction \(x\) of your capital into the risky venture. What is the optimal choice of \(x\), and the corresponding growth rate of your capital, for the various values of \(\theta, u, d, p\)?

Let \(X_0\) be the initial value of the capital. Then, for each \(k\),

\[
X_{k+1} = (1 - x)(1 + \theta)X_k + x(1 + u)X_k,
\]

with probability \(p\) and

\[
X_{k+1} = (1 - x)(1 + \theta)X_k + x(1 - d)X_k,
\]

with probability \(q\). Then, if there are \(Y\) times when the risky investment pays off, we have

\[
X_n = X_0 \left( (1 + \theta)(1 - x) + x(1 + u) \right)^Y \left( (1 + \theta)(1 - x) + x(1 - d) \right)^{n-Y}.
\]

As in the example,

\[
\frac{1}{n} \ln \frac{X_n}{X_0} = \frac{Y}{n} \ln \left( (1 + \theta)(1 - x) + x(1 + u) \right) + \frac{n-Y}{n} \ln \left( (1 + \theta)(1 - x) + x(1 - d) \right)^{n-Y}.
\]

Again, since \(Y/n \to p\) and \((n - Y)/n \to q\) almost surely, we have that

\[
\frac{1}{n} \ln \frac{X_n}{X_0} \to p \ln \left( (1 + \theta)(1 - x) + x(1 + u) \right) + q \ln \left( (1 + \theta)(1 - x) + x(1 - d) \right),
\]

barring divine intervention (i.e. events of probability zero). Considered as a function of \(x\), we wish to maximize this quantity. We will call it \(f(x)\). Then,

\[
f'(x) = \frac{p(u - \theta)}{(1 + \theta)(1 - x) + x(1 + u)} + \frac{q((1 - d) - (1 + \theta))}{(1 + \theta)(1 - x) + x(1 - d)}
\]

\[
= \frac{p(u - \theta)}{1 - x + \theta - \theta x + x + xu} - \frac{q(d + \theta)}{1 - x + \theta - \theta x + x + dx}
\]

\[
= \frac{1}{1 + \theta + x(u - \theta)} - \frac{q(d + \theta)}{1 + \theta - x(d + \theta)}.
\]
If $f(x)$ is a maximum, then $f'(x)$ is zero. This happens when

$$p(u - \theta)(1 + \theta) - x(d + \theta)(u - \theta)p = q(d + \theta)(1 + \theta) + x(u - \theta)(d + \theta)q$$

$$[p(u - \theta) - q(d + \theta)](1 + \theta) = x(d + \theta)(u - \theta)p + x(u - \theta)(d + \theta)q$$

$$= x(d + \theta)(u - \theta)$$

$$x = \frac{p(u - \theta)(1 + \theta) - q(d + \theta)(1 + \theta)}{(u - \theta)(d + \theta)}$$

$$= \frac{p(u - \theta)(1 + \theta) + (p - 1)(d + \theta)(1 + \theta)}{(u - \theta)(d + \theta)}$$

$$= \frac{(1 + \theta)(pu - p\theta + pd + p\theta - d - \theta)}{(u - \theta)(d + \theta)}$$

$$= \frac{(1 + \theta)(pu + pd - d - \theta)}{(u - \theta)(d + \theta)}$$

Since it seems plausible to think that $f$ has a maximum, and since $f'(x)$ is only zero for the value of $x$ given, this is the optimal choice of $x$. Plugging in this value of $x$ into $f$ gives that the optimal growth rate is

$$f(x) = p \ln \left( \frac{u + d}{d + \theta} \right) + q \ln \left( \frac{u + d}{u - \theta} \right).$$

Maple was used for algebraic manipulations in this last step.