1. a) We know that convergence with probability one implies convergence in probability and the reverse implication does not hold. However, show that if $X_n \to X$ in probability as $n \to \infty$, then there is an increasing sequence of positive integers $\{n(k)\}$ such that $X_{n(k)} \to X$ with probability one as $k \to \infty$. That is the sequence $\{X_n\}$ has a subsequence which converges with probability one to $X$.

b) How does this show that there is no metric in which convergence is equivalent to convergence with probability one? Hint: In any metric space, if $\{p_n\}$ is a sequence with the property that every subsequence has a further subsequence which converges to the same limit $p$, then $p_n \to p$.

a) Suppose that $X_n \to X$ in probability. This means that for all $\epsilon > 0$,

$$\lim_{n \to \infty} P(|X_n - X| \geq \epsilon) = 0.$$ 

Then, for all $k \in \mathbb{N}$,

$$\lim_{n \to \infty} P(|X_n - X| \geq 1/k) \to 0.$$ 

Thus, for every $k$, there exists an $n_k$ such that for all $l \geq n_k$

$$P(|X_l - X| \geq 1/k) \leq \frac{1}{2^k}.$$ 

Define $n(k) = n_k$. I claim that $X_{n(k)} \to X$ with probability one as $k \to \infty$. Let

$$S = \{\omega \in \Omega : \lim_{n \to \infty} X_{n(k)}(\omega) = X(\omega)\}.$$ 

Let $T = S^c$. Let

$$T_k = \{\omega : |X_{n(k)}(\omega) - X(\omega)| \geq 1/k\}.$$ 

Note that $P(T_k) \leq 1/2^k$.

Suppose $t \in T$. Then, $\lim_{k \to \infty} X_{n(k)}(\omega) \neq X(\omega)$. This means that there exists $\epsilon > 0$ such that there is an infinite increasing sequence of integers $N = \{n_1, n_2, \ldots\}$ such that $|X_{n(n_i)}(t) - X(t)| > \epsilon$.
Choose a positive integer $k$ such that $k > 1/\epsilon$. Then, there exists an $l$ such that $k < n_l$. Now, $|X_{n_m}(t) - X(t)| > 1/k > 1/n_m$ for all $m \geq l$. and therefore $t \in T_{n_m}$ for all $m \geq l$. Thus, $t$ is in infinitely many of the $T_i$.

Now, let $U$ be the set of points contained in infinitely many of the $T_i$. As shown, $T \subset U$. Note that

$$U = \limsup_{n \to \infty} T_n = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty T_k.$$ 

Now, since

$$\sum_{k=1}^\infty P(T_k) = \sum_{k=1}^\infty \frac{1}{2^k} = 1 < \infty,$$

by the Borel-Cantelli lemmas, it follows that $P(U) = 0$. Now, since $T \subset U$, and $P(U) = 0$, $P(T) = 0$, and therefore $P(S) = 1$, as desired.

b) First, I will prove the hint. Suppose that $\{p_n\}$ is a sequence with the property that every subsequence has a further subsequence that converges to the same limit $p$. Suppose to the contrary that $\lim_{n \to \infty} p_n \neq p$. That means that there exists $\epsilon > 0$ such that for any $N \in \mathbb{N}$, there exists $n > N$ such that $d(p_n, p) > \epsilon$. This means that there exists an infinite sequence $\{n_1, n_2, \ldots\}$ such that $d(p_{n_i}, p) > \epsilon$ for all $i$. This subsequence does not contain a further subsequence that converges to $p$, a contradiction. Hence,

$$\lim_{n \to \infty} p_n = p,$$

as desired.

Now, suppose to the contrary that there were a metric in which the two types of convergence were the same. Then, as mentioned in the first part, there exists a sequence of random variables $X_n$ such that $X_n \to X$ converges in probability but $X_n$ does not converge to $X$ with probability 1. Suppose that $N = \{n_1, n_2, n_3, \ldots\}$ is a subsequence. Then, since for all $\epsilon_1, \epsilon_2 > 0$, there exists an $N$ such that for all $n \geq N$,

$$P(|X_n - X| \geq \epsilon_1) < \epsilon_2.$$ 

Then, if we take $l$ such that $n_l > N$, then for all $m \geq l$,

$$P(|X_{n_m} - X| \geq \epsilon_1) < \epsilon_2.$$ 

Hence, $X_{n_l} \to X$ in probability. Thus, from part a), there is a further subsequence, $K = \{n_1, n_2, \ldots\}$ that converges to $X$ with probability 1. Thus, every subsequence of $\{X_1, X_2, \ldots\}$ contains a further subsequence that converges to $X$ with probability 1. Under the assumption that the convergence is governed by a metric, this implies that the $X_n$s converge to $X$ with probability 1, a contradiction.