Let $F$ be a distribution function and let $R = F^{-1}$ be the corresponding inverse or percentile function.

1. Let $p \in (0, 1)$. Then $R(p) = \min\{x : F(x) \geq p\}$. So, by definition, $F(R(p)) \geq p$. Also, if $x \in \mathbb{R}$, then if we let $q = F(x) \in (0, 1)$, we see that $R(q)$, being the smallest number for which the value of $F$ is greater than or equal to $q = F(x)$, cannot exceed $x$. In other words, $R(F(x)) \leq x$. How should we define $R(p)$ if $p = F(x) = 1$ for some $x$ or $p = F(x) = 0$ for some $x$? Will these inequalities still hold?

2. First suppose that $p \in (0, 1)$ and $x = R(p)$. Then, since we always have $R(F(x)) \leq x$, we see that $R(F(R(p))) \leq R(p)$. Also, $p \leq F(R(p))$ and $R$ is monotone non-decreasing, so that $R(p) \leq R(F(R(p)))$. Consequently, we have $R(F(R(p))) = R(p)$. On the other hand, let $x \in \mathbb{R}$ with $p = F(x)$. Assuming you can define $R(p)$ for $p = 1$ and $p = 0$ in such a way that the inequalities above hold in these cases as well as $p \in (0, 1)$, we have $F(x) \leq F(R(F(x)))$ and $R(F(x)) \leq x$. Since $F$ is monotone non-decreasing, the second inequality shows that $F(R(F(x))) \leq F(x)$. Putting this together, we get $F(R(F(x))) = F(x)$.

3. Suppose that $X$ is a random variable with distribution function $F$, i.e. $X \sim F$. We claim that, even if $F$ is not continuous (which means $F(X)$ cannot be uniformly distributed on $(0, 1)$), $R(F(X)) \sim X$. To see this, note that $X \sim R(U)$, where $R = F^{-1}$ and $U \sim U(0, 1)$. Therefore, we use our result $R(F(R(p))) = R(p)$ to give

$$P(R(F(X))) \leq x) = P(R(F(R(U))) \leq x) = P(R(U) \leq x) = P(X \leq x)$$

for every $x \in \mathbb{R}$. In other words, $R(F(X)) \sim X$. 
