Assignment Sheet#1

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Spring, 2005

Week of March 7, 2005: Measure Spaces and Events

- **Reading:** Begin with the Preface, A Question of Terminology, and A Guide to Notation. Next, read through Chapter 1, noting that Sections 1.0 through 1.5 should be mostly familiar definitions (familiar, for example, from Math 157 handouts: “Review of Definitions of Probability”, “Important Results from Analysis”). Sections 1.6 through 1.8 are the new and/or important material. Sections 1.9 and 1.10 should again be relatively familiar. Section 1.11 is a good example of the danger in interchanging the order of operations. Proofs of the important results in Chapter 1 are found in Appendix A.1. In class we will concentrate on the proof of Uniqueness Lemma 1.6, Carathéodory’s Extension Theorem 1.7, and the construction of Lebesgue measure on (0,1). For Friday, you need to read Chapter 2 on events. Much of this will be familiar (partly from Math 157 handouts: “Important Results from Analysis” and “The Borel-Cantelli Lemmas”).

- **Problems to hand in:** Due on Friday, March 11, 2005.

1. Let $C$ be a collection of subsets of $\Omega$. Show that there is a “smallest” $\sigma$-algebra of subsets of $\Omega$, denoted by $\sigma(C)$, which contains all of the sets which belong to $C$. Be sure you define what you mean by “smallest”.

2. Exercise E1.1 in Chapter E, Exercises, defines the collection $CES$ of subsets of the positive integers $\mathbb{N}$ which have a Cesàro density $\gamma$. Before you work on this exercise, which shows that $CES$ is not an algebra of subsets of $\mathbb{N}$, show that $CES$ is, in fact, closed under complementation, finite disjoint unions, and proper differences (i.e. if $A$ and $B$ are in $CES$ with $A \subseteq B$, then $B \setminus A \in CES$). Moreover, show that the collection $A$ consisting of all finite and co-finite (i.e. the complement is finite) subsets of $\mathbb{N}$ is an algebra contained in $CES$ in which all densities $\gamma$ are either 0 or 1. Hint for the actual exercise in the text: First find a set of positive integers which does not have a Cesàro density. Then write this set as the intersection of two sets which do have densities.
3. (a) Suppose that \( \mathcal{M} \) is the collection of all \textit{bounded} intervals in \( \mathbb{R}^n \), even of some specific type such as open, closed, open on the left and closed on the right, and even if the endpoints are restricted to some dense subset. Show that \( \mathcal{M} \) generates the Borel sets of \( \mathbb{R}^n \).

(b) Suppose that \( \mathcal{M} \) is the set of all intervals of the form

\[
\{ \vec{t} \in \mathbb{R}^n : t_1 \leq a_1, t_2 \leq a_2, \ldots, t_n \leq a_n \} = \times_{j=1}^{n} (-\infty, a_j],
\]

even if the endpoints are restricted to some dense subset. Show that \( \mathcal{M} \) generates the Borel sets of \( \mathbb{R}^n \).

4. Exercise 2.9 in Chapter 2

5. Let \((\Omega, \mathcal{E}, P)\) be a probability space (a probability triple in the terminology of your text) and let \( \{A_n\} \) be a sequence in \( \mathcal{E} \).

(a) Prove that if \( \limsup_{n \to \infty} P(A_n) > 0 \), then \( \limsup_{n \to \infty} A_n \) is not empty.

(b) In particular, explain why this result shows (assuming knowledge of Lebesgue measure) that if we are given a sequence of subintervals of \([0, 1]\), all of which have length at least \( \delta > 0 \), then there are an uncountable number of points which belong to an infinite number of these subintervals. Now, give an elementary (i.e. no measure theory allowed) proof of this result.

- **Spring Break Reading:** Read Chapter 0 to get a preview of where measure theory is essential in a more sophisticated analysis of the standard model for a branching process

- **Extra Credit Problems:** Look at problems EG.2, EG.3, and EG.4 in Chapter E, Exercises. Correct solutions to any of these completely non-measure theoretic probability problems submitted any time before the end of the course will win you a big bonus.