Assignment Sheet #4

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Spring, 2005

Week of April 11: Integrals and Expected Values

• **Reading:** Chapter 5 and Chapter 6 should be read along with the Appendix to Chapter 5, namely Chapter A5. In particular, in Chapter 5, look at the Note on the Riemann Integral on page 52, Section 5.12 on the “Standard Machine” and Section 5.14 where the Radon Nikodym Theorem is stated. In Chapter 6, which reviews many of the concepts from Intermediate Probability in a more general context, be sure to read carefully Section 6.6 on Jensen’s inequality, Section 6.10 on the completeness of $L^p$ for $1 \leq p < \infty$, and Section 6.12 which shows that expected value as an integral agrees with the versions of expected value you have studied in Intermediate Probability.

• **Problems to hand in:** Due on Monday, April 18, 2005 at 5:00 PM.

3. Prove the second part of Scheffé’s Lemma (page 55), namely: Suppose that $f_n, f \in L^1(S, \Sigma, \mu)$ and that $f_n \to f$ (a.e.). Then $\mu(|f_n - f|) \to 0$ if and only if $\mu(|f_n|) \to \mu(|f|)$.
   Hint: follow the outline given in the book.
4. Let $G$ be an open subinterval of $\mathbb{R}$ and let $c : G \to \mathbb{R}$ be convex.
   (a) Use the characterization of convexity given on page 61, namely $\Delta_{u,v} \leq \Delta_{v,w}$ for $u, v, w \in G$ with $u < v < w$, to show that $c$ is continuous on $G$.
   Hint: show that for each point $y \in G$, $\limsup_{n \to \infty} c(x_n) \leq c(y)$ and $\liminf_{n \to \infty} c(x_n) \geq c(y)$ whenever $\{x_n\}$ is a sequence in $G$ such that $x_n \uparrow y$. Then do the same thing when $x_n \downarrow y$.
   (b) Prove the assertion of the Remark near the bottom of page 61, namely: there exist sequences $\{a_n\}$ and $\{b_n\}$ in $\mathbb{R}$ such that for all $x \in G$ we have
   $$c(x) = \sup_{q \in G} [(D_- c)(q)(x - q) + c(q)] = \sup_n (a_n x + b_n).$$