Question 2. Let $n_0 = 0$ and, for $k \geq 1$, let $n_{2k-1} = 2 \cdot 2^k$, $n_{2k} = 3 \cdot 2^k$. Then $n_0 < n_1$ and, for $k \geq 1$, $n_{2k-1} < n_{2k} < n_{2k+1}$. Moreover, for $k \geq 1$,

$$n_{2k} - n_{2k-1} = n_{2k+1} - n_{2k} = 2^k.$$

Now define $A \subset \mathbb{N}$ as follows: if $m \in (n_{2k-1}, n_{2k}]$ for some $k \geq 1$, let $m \in A$ iff $m$ is even, and, for each $k \geq 0$, $A \cap (n_{2k}, n_{2k+1}] = \emptyset$.

We claim that $A$ does not have a Cesaro density. To see this note that for $k \geq 1$,

$$|A \cap (0, n_{2k}]| = |A \cap (0, n_{2k+1}]| = \sum_{j=1}^{k} 2^{j-1} = 2^k - 1.$$

Therefore,

$$\limsup_{n \to \infty} \frac{|A \cap (0, n_k]|}{n} \geq \lim_{k \to \infty} \frac{|A \cap (0, n_{2k}]|}{n_{2k}} = \lim_{k \to \infty} \frac{2^k - 1}{3 \cdot 2^k} = \frac{1}{3}.$$

On the other hand,

$$\liminf_{n \to \infty} \frac{|A \cap (0, n_k]|}{n} \leq \lim_{k \to \infty} \frac{|A \cap (0, n_{2k+1}]|}{n_{2k+1}} = \lim_{k \to \infty} \frac{2^k - 1}{2 \cdot 2^{k+1}} = \frac{1}{4}.$$

Now define sets $V_1$ and $V_2$ as follows: for $m \in (n_{2k-1}, n_{2k}]$, $k \geq 1$, let $V_1$ and $V_2$ agree with $A$, i.e. the even integers in these intervals. For $m \in (n_{2k}, n_{2k+1}]$, $k \geq 0$, let $m \in V_1$ iff $m$ is even and $m \in V_2$ iff $m$ is odd. Then $A = V_1 \cap V_2$. Moreover, for all even $n$, we have

$$\frac{|V_1 \cap (0, n]|}{n} = \frac{|V_2 \cap (0, n]|}{n} = \frac{1}{2}.$$

On the other hand, if $n$ is odd, we see that

$$\frac{1}{2} - \frac{1}{2n} \leq \frac{|V_1 \cap (0, n]|}{n} \leq \frac{1}{2}$$

and

$$\frac{1}{2} - \frac{1}{2n} \leq \frac{|V_2 \cap (0, n]|}{n} \leq \frac{1}{2} + \frac{1}{2n}.$$

Therefore, both $V_1$ and $V_2$ have Cesaro density $1/2$, so they each belong to $CES$. However, their intersection $A$ does not belong to $CES$ and thus $CES$ is not an algebra of subsets of $\mathbb{N}$. 
