The Borel-Cantelli Lemmas

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Let \( \{A_n\} \) be a sequence of events. Then

\[ \{A_n \ i.o.\} = \limsup_{n \to \infty} A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k \]

is the event which occurs if and only if an \textit{infinite} number of the events \( A_n \)
occur. The \textit{i.o.} stands for “infinitely often”. Similarly,

\[ \liminf_{n \to \infty} A_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k \]

is the event which occurs if and only if \textit{all but a finite} number of the events \( A_n \)
occur. Note that

\[ \liminf_{n \to \infty} A_n \subset \limsup_{n \to \infty} A_n. \]

We say that the sequence \( \{A_n\} \) has a \textit{limit} if and only if these two events are equal. One situation in which this occurs is when \( \{A_n\} \) is \textit{monotone}. In the case of monotone increasing \( \{A_n\} \), the limit is \( \cup_{n=1}^{\infty} A_n \). In the case of monotone decreasing \( \{A_n\} \), the limit is \( \cap_{n=1}^{\infty} A_n \). In any case in which \( \{A_n\} \) has a \textit{limit},
call it \( A \), we claim that

\[ \lim_{n \to \infty} P(A_n) = P(A). \]

To see this, note that

\[ P(\liminf_{n \to \infty} A_n) = \lim_{n \to \infty} P(\cap_{k=n}^{\infty} A_k) \leq \liminf_{n \to \infty} P(A_n) \]

and

\[ \limsup_{n \to \infty} P(A_n) \leq \lim_{n \to \infty} P(\cup_{k=n}^{\infty} A_k) = P(\limsup_{n \to \infty} A_n). \]

\textbf{Borel-Cantelli Lemma:} If \( \{A_n\} \) is a sequence of events and

\[ \sum_{n=1}^{\infty} P(A_n) < \infty, \]

then

\[ P(\{A_n \ i.o.\}) = 0. \]
Proof:

\[ P(\{A_n \ i.o.\}) = P(\limsup_{n \to \infty} A_n) = \lim_{n \to \infty} P(\bigcup_{k=n}^{\infty} A_k) \leq \lim_{n \to \infty} \sum_{k=n}^{\infty} P(A_k) \]

and this tail sum approaches 0 as \( n \to \infty \) under the assumption that the series converges.

Converse to Borelli-Cantelli Lemma: Now suppose that \( \{A_n\} \) is a sequence of independent events and

\[ \sum_{n=1}^{\infty} P(A_n) = \infty, \]

then

\[ P(\{A_n \ i.o.\}) = 1. \]

Proof: First note that

\[ 1 - P(\{A_n \ i.o.\}) = P(\{A_n \ i.o.\}^c) = P(\liminf_{n \to \infty} A_n^c) = \lim_{n \to \infty} P(\bigcap_{k=n}^{\infty} A_k^c). \]

Because of independence of the \( \{A_n\} \) and hence of the \( \{A_n^c\} \), we see that for every \( n \),

\[ P(\bigcap_{k=n}^{\infty} A_k^c) = \lim_{N \to \infty} P(\bigcap_{k=n}^{N} A_k^c) = \lim_{N \to \infty} \prod_{k=n}^{N} P(A_k^c) \leq \lim_{N \to \infty} \prod_{k=n}^{N} [1 - P(A_k)] \]

\[ = \lim_{N \to \infty} \prod_{k=n}^{N} \exp[-P(A_k)] \]

\[ = \lim_{N \to \infty} \exp \left[ - \sum_{k=n}^{N} P(A_k) \right] \]

\[ = 0, \]

since the series diverges and the exponential function approaches 0 as the exponent approaches \( -\infty \).