Instructions:
This is a practice test for an in-class examination which will be held during your class period on Friday, November 14. The actual exam is closed-book and closed-notes, except for a one-sided 8.5" × 11" sheet of notes which, if used, must be submitted with your exam. Calculators are permitted. Since 10% of your score on the actual exam comes from your up to one page essay (which also must be submitted with your exam) related to a current newspaper or other periodical that involves concepts from probability and/or statistics, the number of points for questions on the exam adds up to 90. The material covered on the exam is based on homework assignments 1 through 6.
Questions

1. (10 points) T or F: Mark T (true) if the statement is always true; mark F (false) if the statement is sometimes false.

   a) Data are always categorical.
   b) Outcomes are subsets of the sample space.
   c) The total number of subsets of a set of size \( N \) is \( 2^N \).
   d) \( P(A \cup B \cup C) \geq P(C) \)
   e) \( P(B^c) = 1 - P(B) \)
   f) If \( A \) and \( B \) are independent, then \( P(A \cap B) = P(A)P(B) \).
   g) If \( A \) and \( B \) are mutually exclusive, then \( A \cap B = \emptyset \).
   h) The number of trials until the first success in a sequence of independent Bernoulli(p) trials has a geometric distribution.
   i) \( E(aX + b) = aE(X) + b \)
   j) \( Var(aX + b) = a^2Var(X) + b^2 \)

Solution:

a) F
b) F
c) T
d) T
e) T
f) T
g) T
h) T
i) T
j) F
2. (20 points) Suppose 3 bagels are picked at random from a bag containing 4 cinammon raisin, 3 blueberry, and 2 plain bagels. What is the probability that:

(a) A plain bagel is among the bagels selected?
(b) 2 cinammon raisin and 1 blueberry are selected?

Solution:

(a) Let \( X \) denote the number of plain bagels selected. Then,

\[
P(X \geq 1) = P(X = 1) + P(X = 2) = 1 - P(X = 0).
\]

However,

\[
P(X = 0) = \frac{C_7^3}{C_9^3} = \frac{7 \cdot 6 \cdot 5}{9 \cdot 8 \cdot 7} = \frac{5}{12}.
\]

Therefore, \( P(X \geq 1) = \frac{7}{12} \).

(b) Let \( Y \) denote the number of cinammon raisin bagels selected and let \( Z \) denote the number of blueberry bagels selected. Then,

\[
P(Y = 2, Z = 1) = \frac{C_4^2 C_1^3}{C_3^3} = \frac{6 \cdot 3}{(9 \cdot 8 \cdot 7)/6} = \frac{3}{14}.
\]
3. (20 points) The probability that a chemical reaction succeeds at room temperature is 0.75.

(a) One fine afternoon, this reaction is attempted by 18 students in a chemistry lab section. What is the probability that at least 15 of their reactions succeed? What assumptions did you make when you performed this calculation?

(b) During the entire week, 90 students attempted the reaction. What is the expected number of failures?

Solution:

(a) Assuming the experiments are performed independently (and with the same probability of success), then if we let $X$ denote the number of students who succeeded, $X$ has a binomial distribution with $n = 18$ and $p = .75$. Therefore,

$$P(X \geq 15) = P(X = 15) + P(X = 16) + P(X = 17) + P(X = 18)$$

$$= \binom{18}{15}(.75)^{15}(.25)^3 + \binom{18}{16}(.75)^{16}(.25)^2$$

$$+ \binom{18}{17}(.75)^{17}(.25) + (.75)^{18}$$

$$= .1704 + .0958 + .0338 + .0056$$

$$= .3056.$$

(b) If $Y$ is the number of failures in $n = 90$ trials with probability of failure on each trial $q = .25$, then the expected number of failures (whether or not these trials are independent) is

$$E(N) = nq = (90)(.25) = 22.5.$$
4. (20 points)

(a) The time between hits on a popular website is described by a random variable with an exponential($\lambda$) distribution, where the rate parameter $\lambda$ is 100 hits per minute. Show that the mean time between hits for this exponential distribution is .01 minutes.

(b) Find the probability that the time between hits is greater than 1 second.

(c) Find the expected number of gaps between successive hits until the first gap which is greater than 1 second. What assumptions did you make to perform this calculation?

Solution:

(a) If $X$ has an exponential distribution with parameter $\lambda = 100$ hits per minute, then

$$E(X) = \int_0^\infty xf_X(x) \, dx = \int_0^\infty x\lambda e^{-\lambda x} \, dx = \frac{1}{\lambda}.$$ 

Therefore, $E(X) = (1/100)\text{minute} = .01\text{minute}.$

(b) Since for $x > 0$,

$$P(X > x) = e^{-\lambda x},$$

we can see that when $x = 1\text{sec} = 1/60\text{minute},$

$$P(X > 1/60) = e^{-100/60} = e^{-5/3} = .1889.$$

(c) Assuming the gaps, i.e. the times between successive hits, are independent, we have a sequence of independent Bernoulli trials with common probability of “success”, namely a gap exceeding 1 second, $p = .1889$. Therefore, if $Y$ is the number of gaps until the first success, $Y$ has a geometric distribution with this parameter $p$ and $E(Y) = 1/p = 5.2945$. 


5. (20 points) A plant manufacturing microprocessors has two shifts: shift I and shift II. It is known that shift I produces 60 percent of the microprocessors made in the plant and that 3 percent of its production will not meet customer specifications. Similarly, 5 percent of shift II’s production will not meet customer specifications.

(a) What percent of the plant’s production will not meet customer specifications?

(b) A microprocessor is chosen at random and found not to meet customer specifications. What is the probability that it was produced by shift I?

Solution:

(a) Let $SI$ and $SII$ represent the events that a microprocessor is produced by a particular shift, so that without any other information we know $P(SI) = .60$ and $P(SII) = .40$. We also know that if $D$ represents the event that a defective (not meeting customer specifications) microprocessor is produced, $P(D|SI) = .03$ and $P(D|SII) = .05$.

Therefore, by the law of total probability,

$$P(D) = P(D|SI)P(SI) + P(D|SII)P(SII) = (.03)(.60) + (.05)(.40) = .038$$

(b) The probability that a microprocessor chosen at random and found to be defective was produced by shift $SI$ is given by

$$P(SI|D) = \frac{P(SI \cap D)}{P(D)} = \frac{P(D|SI)P(SI)}{P(D)} = \frac{.018}{.038} = \frac{9}{19}.$$

Note that this updated probability is less than 1/2 and considerably less than the unconditional probability $P(SI) = .60$. 