Verifying the Markov Property in Continuous Time

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Review: Let \( \{X_n : n = 0, 1, 2, \ldots \} \) be a discrete time Markov chain on the finite or countably infinite state space \( \mathcal{S} \) with transition probability matrix \( Q \) such that \( Q(x, x) = 0 \) for all \( x \in \mathcal{S} \). For each \( x \in \mathcal{S} \), let \( q_x \geq 0 \) be the holding time parameter in the state \( x \) (i.e. the probability of remaining in state \( x \) for an amount of time greater than \( t \) is \( e^{-q_x t} \), so that \( 1/q_x \) is the mean of the exponentially distributed holding time in state \( x \) if \( q_x > 0 \) and \( q_x = 0 \) whenever \( x \) is an absorbing state.) Let \( \{E_n : n = 0, 1, 2, \ldots \} \) be a sequence of independent, identically distributed exponential random variables, each with mean 1, and independent of the \( \{X_n\} \).

Definitions: Let \( T_0 = 0 \) and for \( n \geq 1 \) let

\[
T_n = \sum_{k=0}^{n-1} \frac{E_k}{q_{X_k}}.
\]

We then define, for \( 0 \leq t < T_\infty = \lim_{n \to \infty} T_n \),

\[
X(t) = \sum_{n=0}^{\infty} X_n \mathbf{1}_{[T_n, T_{n+1})}(t).
\]

Finally, for \( x \) and \( y \) in \( \mathcal{S} \) and \( 0 \leq t < T_\infty \), we define the transition probability function

\[
P_{x,y}(t) = P(X(t) = y | X(0) = x) = P_x(X(t) = y).
\]

Theorem: If \( x, y_1, \) and \( y_2 \) belong to \( \mathcal{S} \) and \( 0 < t_1 < t_2 < T_\infty \), then

\[
P_x(X(t_1) = y_1, X(t_2) = y_2) = P_{x,y_1}(t_1)P_{y_1,y_2}(t_2 - t_1).
\]

Proof: We first note that we can write

\[
P_{x,y}(t) = \sum_{n=0}^{\infty} P_x(X_n = y, T_n \leq t < T_{n+1}).
\]

Now let's examine the terms in this sum. For \( n = 0 \),

\[
P_x(X_0 = y, T_0 \leq t < T_1) = \delta_{x,y} e^{-q_x t}.
\]
On the other hand, if \( n \geq 1 \),

\[
P_x(X_n = y, T_n \leq t < T_{n+1})
\]

is the sum over \( x_1, \ldots, x_{n-1} \in S \) of

\[
P_x(X_j = x_j, 1 \leq j < n, X_n = y, T_n \leq t < T_{n+1}),
\]

where

\[
P_x(X_j = x_j, 1 \leq j < n, X_n = y, T_n \leq t < T_{n+1})
\]

is the product of

\[
P_x(T_n \leq t < T_{n+1} | X_j = x_j, 1 \leq j < n, X_n = y)
\]

and

\[
P_x(X_j = x_j, 1 \leq j < n, X_n = y).
\]

Thus if \( F = F_{x,x_1,\ldots,x_{n-1}} \) is the distribution function of

\[
E_0/q_x + E_1/q_{x_1} + \cdots + E_{n-1}/q_{x_{n-1}},
\]

i.e. the distribution function of \( T_n \) given \( X_0 = x, X_1 = x_1, \ldots, X_{n-1} = x_{n-1} \), we see that

\[
P_x(T_n \leq t < T_{n+1} | X_j = x_j, 1 \leq j < n, X_n = y)
\]

\[
= \int_0^t P_x(T_n \leq t < T_{n+1} | X_j = x_j, 1 \leq j < n, X_n = y, T_n = s) dF(s)
\]

\[
= \int_0^t P(E_n/y > t - s) dF(s)
\]

\[
= \int_0^t \int_{t-s}^{\infty} q_y e^{-q_y u} du dF(s)
\]

\[
= \int_0^t e^{-q_y (t-s)} dF_{x,x_1,\ldots,x_{n-1}}(s).
\]

Note that the expression for \( n = 0 \) can be interpreted as a special case of the above if we let the distribution function \( F \) be concentrated at 0. Moving to the case of two time points, we can write

\[
P_x(X(t_1) = y_1, X(t_2) = y_2) =
\]

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_x(X_n = y_1, T_n \leq t_1 < T_{n+1}, X_{n+m} = y_2, T_{n+m} \leq t_2 < T_{n+m+1}).
\]

Thus once again we see that

\[
P_x(X_n = y_1, T_n \leq t_1 < T_{n+1}, X_{n+m} = y_2, T_{n+m} \leq t_2 < T_{n+m+1})
\]

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can be written as the sum over \( x_1, \ldots, x_{n-1}, \ldots, x_{n+m-1} \in \mathcal{S} \) of
\[
P_x(X_1 = x_1, \ldots, X_n = y_1, \ldots, X_{n+m} = y_2, T_n \leq t_1 < T_{n+1}, T_{n+m} \leq t_2 < T_{n+m+1}),
\]
where
\[
P_x(X_1 = x_1, \ldots, X_n = y_1, \ldots, X_{n+m} = y_2, T_n \leq t_1 < T_{n+1}, T_{n+m} \leq t_2 < T_{n+m+1})
\]
is the product of
\[
P_x(T_n \leq t_1 < T_{n+1}, T_{n+m} \leq t_2 < T_{n+m+1}|X_1 = x_1, \ldots, X_n = y_1, \ldots, X_{n+m} = y_2).
\]
Using the Markov property for the chain \( \{X_n\} \), we see that
\[
P_x(X_1 = x_1, \ldots, X_n = y_1, \ldots, X_{n+m-1} = x_{n+m-1}, X_{n+m} = y_2)
\]
is the product of
\[
P_x(X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_n = y_1)
\]
and
\[
P_y(x_1, \ldots, X_{m-1} = x_{m-1}, X_m = y_2).
\]
Moreover, again letting \( F = F_{x_1, \ldots, x_{n-1}} \) be the conditional distribution of
\[
P_x(T_n \leq t_1 < T_{n+1}, T_{n+m} \leq t_2 < T_{n+m+1}|X_1 = x_1, \ldots, X_n = y_1, \ldots, X_{n+m} = y_2)
\]
can be written as
\[
\int_0^{t_1} \int_{s-u}^\infty q_{y_1} e^{-q_{y_1} u} P_x \left( s + u + \sum_{k=n+1}^{n+m-1} \frac{E_k}{q_{x_k}} \leq t_2 < s + u + \sum_{k=n+1}^{n+m-1} \frac{E_k}{q_{x_k}} \frac{E_{n+m}}{q_{y_2}} \right) du \, dF(s).
\]
If we make the change of variable, \( v = u - (t_1 - s) = s + u - t_1 \), we get
\[
\int_0^{t_1} \int_0^\infty q_{y_1} e^{-q_{y_1} (v+t_1-s)} P \left( v + \sum_{k=n+1}^{n+m-1} \frac{E_k}{q_{x_k}} \leq t_2 - t_1 < v + \sum_{k=n+1}^{n+m-1} \frac{E_k}{q_{x_k}} \frac{E_{n+m}}{q_{y_2}} \right) dv \, dF(s),
\]
which is the product of two integrals,
\[
\int_0^{t_1} e^{-q_{y_1} (t_1-s)} dF(s)
\]
and
\[
\int_0^\infty q_{y_1} e^{-q_{y_1} v} P \left( v + \sum_{k=n+1}^{n+m-1} \frac{E_k}{q_{x_k}} \leq t_2 - t_1 < v + \sum_{k=n+1}^{n+m-1} \frac{E_k}{q_{x_k}} \frac{E_{n+m}}{q_{y_2}} \right) dv.
\]
However, the first of these integrals is just

\[ P_x(T_n \leq t_1 < T_{n+1} | X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_n = y_1) \]

and the second is

\[ P_{y_1}(T_m \leq t_2 - t_1 < T_{m+1} | X_1 = x_{n+1}, \ldots, X_{m-1} = x_{n+m-1}, X_m = y_2). \]

Consequently, putting these pieces together (by multiplying the conditional probabilities by the events we conditioned on and then summing these results), we get the desired result

\[ P_x(X(t_1) = y_1, X(t_2) = y_2) = P_{x,y_1}(t_1)P_{y_1,y_2}(t_2 - t_1). \]