Exercises on Distribution and Percentile Functions

H. Krieger, Mathematics 156, Harvey Mudd College

Fall, 2008

Throughout these exercises, unless otherwise specified, we let \( F \) be a distribution function and \( R = F^{-1} \) be the corresponding percentile function.

1. If \( X \) has an exponential distribution with mean \( \mu \), find \( F_X \) and \( R_X \) explicitly.

2. Show that for arbitrary distribution function \( F \), the percentile function \( R \) is always non-decreasing on \((0, 1)\) and continuous from below.

3. Show that \( p \leq F(R(p)) \) for every \( p \in (0, 1) \) and that \( R(F(x)) \leq x \) for every \( x \in \mathbb{R} \).

4. Consequently, show that \( R(F(R(p)))) = R(p) \) for every \( p \in (0, 1) \) and that \( F(R(F(x))) = F(x) \) for every \( x \in \mathbb{R} \).

5. Show that if \( X \sim F \) then, even if \( F \) is not continuous, \( R(F(X)) \sim X \). Hint: \( X \sim R(U) \) where \( U \sim U(0, 1) \).

6. For an explicit example of this, let

\[
P(X = -1) = 1/3 \text{ and } P(X = 2) = 2/3.
\]

Sketch the graphs of the corresponding distribution function \( F \) and percentile function \( R \). Then verify that \( X \sim R(U) \), where \( U \sim U(0, 1) \), and \( X \sim R(F(X)) \).

7. From above we know that if \( p = F(x) \) for some \( x \in \mathbb{R} \), i.e. \( p \) is in the range of \( F \), then \( F(R(p)) = p \). Show that this condition implies that \( p \) is a point of increase of \( R \). Here, \( p \) is a point of increase of \( R \) if and only if for every \( \varepsilon > 0 \), \( R(p + \varepsilon) - R(p - \varepsilon) > 0 \). Consequently, show that every \( p \in (0, 1) \) is a point of increase of \( R \) if \( F \) is continuous.

8. Similarly, we know that if \( x = R(p) \) for some \( p \in (0, 1) \), i.e. \( x \) is in the range of \( R \), then \( R(F(x)) = x \). Show that this condition implies that \( x \) is a point of increase of \( F \).
9. Suppose that \( R \) is a non-decreasing function on \((0, 1)\) which is continuous from below. We can extend \( R \) to \([0, 1]\) by taking limits from within \((0, 1)\) at the endpoints. To define an inverse for \( R \), we proceed as follows. If \( x < R(0) \) define \( R^{-1}(x) = 0 \). Otherwise, let \( R^{-1}(x) = \max\{ p : R(p) \leq x \} \). Show that \( R^{-1} \) is a distribution function. Also, show that if \( R = F^{-1} \) for some distribution function \( F \), then \( R^{-1} = F \).

10. A four-parameter family of distributions can be defined for certain values of these parameters by the percentile function

\( R(p) = \lambda_1 + [p^{\lambda_3} - (1 - p)^{\lambda_4}] / \lambda_2 \)

for \( p \in (0, 1) \). Here, \( \lambda_1 \) is a location parameter, \( \lambda_2 \neq 0 \) is a scale parameter, and \( \lambda_3 \) and \( \lambda_4 \) are shape parameters (they determine skewness and kurtosis).

(a) Since \( R \) is differentiable, first show that

\( R'(p) = [\lambda_3 p^{\lambda_3 - 1} + \lambda_4 (1 - p)^{\lambda_4 - 1}] / \lambda_2. \)

Then show that \( R \) is strictly increasing on \((0, 1)\) if and only if \( \lambda_3 \lambda_4 \geq 0 \), \( \lambda_3 + \lambda_4 \neq 0 \), and \( \text{sign}(\lambda_2) = \text{sign}(\lambda_3 + \lambda_4) \).

(b) Assuming these conditions are satisfied, plot the density function (parametrically, using e.g. Maple, Matlab, Mathematica) for the distribution with \( \lambda_1 = 0 \), \( \lambda_2 = 0.1975 \), and \( \lambda_3 = \lambda_4 = 0.1349 \).

Compare this plot with that of the standard normal density.

(c) Now suppose that \( X \sim R(U) \) where \( U \sim U(0, 1) \). Then, assuming that these moments exist, show that

\[
E(|X - \lambda_1|^k) = \int_0^1 |R(p) - \lambda_1|^k dp = \frac{\lambda_2^k}{A_k},
\]

where

\[
A_k = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \beta(1 + (k - j)\lambda_3, 1 + j\lambda_4).
\]

Here, the beta function is defined for \( x > 0 \) and \( y > 0 \) by

\[
\beta(x, y) = \int_0^1 p^{x-1} (1 - p)^{y-1} dp = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)},
\]

where the gamma function (generalized factorial) is defined for \( x > 0 \) by

\[
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.
\]

This function has the properties \( \Gamma(1) = 1 \) and \( x\Gamma(x) = \Gamma(x + 1) \) so that \( \beta(x, y) = \beta(y, x) \) and \( \beta(x, 1) = 1/x \).
(d) Show that when $k$ is a positive integer, $A_k$ exists if $\min\{\lambda_3, \lambda_4\} > -1/k$.

(e) Evaluate $A_k$ for $k = 0, 1, 2$. Also, show that $m_1 = E(X) = \lambda_2^{-1} A_1 + \lambda_1$ and $\sigma^2 = \mu_2 = E([X - m_1]^2) = \lambda_2^{-2} (A_2 - A_1^2)$.

(f) Show that $A_2 - A_1^2 \geq 0$ and thus we can solve to obtain $\lambda_2 = \lambda_2^* / \sigma$, where $\lambda_2^* = \text{sign}(\lambda_3 + \lambda_4) \sqrt{A_2 - A_1^2}$ is the value of $\lambda_2$ assuming $\sigma^2 = 1$.

(g) Similarly, if $\lambda_1^*$ is the value of $\lambda_1$ assuming $m_1 = 0$ and $\sigma^2 = 1$, $i.e.$ $\lambda_1^* = -A_1 / \lambda_2^*$, show that $\lambda_1 = m_1 + \lambda_1^* \sigma$. 

3