2 The Fractional Chromatic Gap

As previously noted, $\chi_f(G) = \omega_f(G)$ for any finite graph $G$. This result follows from the strong duality of linear programs. Since there is no such duality result for infinite linear programs, it is reasonable to ask, “Does this equality still hold?” In general it does not, and the proof of this comes in conjunction with an open problem presented by Leader [8], which shall be addressed after presenting a few definitions.

To start, we need to alter the (LP) formulation of $\chi_f(G)$ to remove the linear programming language and accommodate infinite graphs. In this we follow Leader’s notation. We still let $\mathcal{I}$ represent the set of independent sets of $G$, and then define a fractional coloring of $G$ to be a mapping $f : \mathcal{I} \rightarrow [0, 1]$ such that for each $v \in G$ we have $\sum_{I \in \mathcal{I} : v \in I} f(I) \geq 1$. The weight of this coloring is $w(f) = \sum_{I \in \mathcal{I}} f(I)$. Leader’s definition of fractional chromatic number (which we call $\chi^*(G)$ for now) is

$$\chi^*(G) = \inf \{ w(f) : f \text{ a fractional coloring of } G \}.$$ 

Note that, if $G$ is finite, this is equivalent to the (LP) formulation. Further, $\chi^*(G)$ is well-defined in the case that $G$ is infinite. For the time being, we reserve the $\chi_f$ notation to represent the $b$-fold formulation, which is still clearly valid for infinite graphs. What is not clear is that these two formulations are still equivalent in the infinite case. We shall show that they are.

We similarly modify the definition of fractional clique number: a fractional clique of $G$ is a mapping $g : V(G) \rightarrow [0, 1]$ such that for each $I \in \mathcal{I}$ we have $\sum_{v \in I} g(v) \leq 1$. The weight of this mapping is $w(g) = \sum_{v \in V(G)} g(v)$, and fractional clique number is
\[ \omega^*(G) = \sup \{w(g) : g \text{ a fractional clique of } G\} . \]

Again, this is equivalent to (DP) if \( G \) is finite\(^8\).

Finally, we define

\[ \overline{\chi_f}(G) = \sup \{\chi_f(H) : H \text{ a finite subgraph of } G\} , \]

The Erdős-de Bruijn theorem [4] states that the ordinary chromatic number of an infinite graph equals the supremum of the chromatic numbers of all its finite subgraphs. Zhu [17] asked if this was the case for fractional chromatic number, and Leader [8] answered in the negative by constructing graphs \( G \) with \( \chi_f(G) = \infty \) and \( \overline{\chi_f}(G) \) any rational number larger than 2. He then asked if there exists an infinite \( G \) for which \( \overline{\chi_f}(G) < \chi_f(G) < \infty \).

In this chapter, we construct a class of such graphs. Further, by proving that \( \omega_f(G) = \overline{\chi_f}(G) \), we show that the strong duality result for fractional chromatic number is, in general, false for infinite graphs.

\section*{2.1 Two Equalities for Infinite Graphs}

The proof that \( \chi^*(G) = \lim_{b \to \infty} \chi_b(G)/b \) for finite \( G \) is fairly simple, but depends both on there being only a finite number of independent sets, and on the fact that linear programs with integer coefficients have rational solutions. For infinite graphs, we have neither of these. However, we may find rational and finite fractional colorings arbitrarily close to any fractional coloring, and this is sufficient.

\(^8\)Since we do not use the \( b \)-fold formulation of \( \omega_f \) here, we henceforth use this notation in place of \( \omega^* \). A proof that these two are equal appears in Appendix A.
Theorem 2.1  For any infinite graph $G$, $\chi^*(G) = \chi_f(G)$.

Proof. The proof from Lemma 1.1 that $\chi^*(G) \leq \lim_{b \to \infty} \chi_b(G)/b$ still works in the infinite case\(^9\): any $b$-fold coloring of value $\chi_b(G)$ is transformed into a fractional coloring of value $\chi_b(G)/b$, thereby proving that $\chi^*(G) \leq \chi_f(G)$.

Since $\chi^*(G) \leq \chi_f(G) \leq \chi(G)$, and Leader[8] showed that $\chi(G) = \infty$ implies $\chi^*(G) = \infty$, then we are done if $\chi^*(G) = \infty$. So we restrict our attention to the case of $\chi^*(G) < \infty$.

To proceed with the other inequality, for any given $\epsilon > 0$, we wish to find a positive integer $b_0$ such that $\chi_b(G)/b \leq \chi^*(G) + \epsilon$ for all $b \geq b_0$. If we can accomplish this, then $\lim_{b \to \infty} \chi_b(G)/b \leq \chi^*(G)$, and we are done. We will start with a fractional coloring $f$ of weight sufficiently close to $\chi_f(G)$, then make two approximations of it: (i) restrict $f$ to being positive on only a finite number of independent sets, and (ii) find sufficiently large $b$ so that $f$ may be rounded up to multiples of $1/b$ with negligible addition of total weight.

We may then use the method of Lemma 1.1 to convert $f$ into a proper $b$-fold coloring.

(i) Given $\epsilon > 0$, take $\delta = \frac{\epsilon}{2}(1 + \chi^*(G) + \frac{\epsilon}{2})^{-1}$, so that $\frac{\chi_f(G) + \delta}{1 - \delta} = \chi^*(G) + \epsilon/2$. Now, choose a fractional coloring $f$ with $\chi^*(G) < w(f) \leq \chi^*(G) + \delta$. Since $w(f)$ is a (possibly) infinite sum of finite value, we may find finite partial sums arbitrarily close to this value. More specifically, there exists a finite $\mathcal{I}' \subset \mathcal{I}$ (where $\mathcal{I}$ are the independent sets of $G$) such that

$$\chi^*(G) \leq \sum_{I \in \mathcal{I}'} f(I) \leq \sum_{I \in \mathcal{I}} f(I) = w(f) \leq \chi^*(G) + \delta.$$  

\(^9\)Note, however, that what we called “$\chi_f$” in that proof we are now calling “$\chi^*$”.\n
Let $n = |\mathcal{I}'|$, and let $f'$ be the weighting of $\mathcal{I}$ by $f$ restricted to $\mathcal{I}'$, so that $w(f') = \sum_{I \in \mathcal{I}'} f(I)$. Since $w(f')$ is within $\delta$ of $w(f)$, $f'$ must give each vertex of $G$ weight at least $1 - \delta$. We now define $f''$ by multiplying each $f'(I)$ by $1/(1-\delta)$, so that $f''$ gives each vertex weight at least 1, and is thus a valid fractional coloring. Further, $w(f'') = w(f')/(1-\delta) \leq \frac{\chi^*(G)+\delta}{1-\delta} = \chi^*(G) + \epsilon/2$ by our choice of $\delta$.

(ii) Now, choose $b > 2n/\epsilon$, and create the fractional coloring $f_b$ by rounding $f''(I)$ up to the nearest multiple of $1/b$ for each $I \in \mathcal{I}'$. We have only added weight, so $f_b$ is still a valid fractional coloring, and we have added at most $|\mathcal{I}'|/b = n/b < \epsilon/2$ weight to $w(f'')$. So $w(f_b) \leq w(f'') + \epsilon/2 \leq \chi^*(G) + \epsilon$. We now have a rational fractional coloring with common denominator $b$ using only a finite collection of independent sets. For each $I \in \mathcal{I}'$, if we associate $b \cdot f_b(I)$ distinct colors (and apply them to each $v \in I$), then every vertex in $G$ gets (at least) $b$ colors. Since each color constitutes an independent set, we have created a proper $b$-fold coloring. We have used $b \cdot w(f_b)$ colors, and so $\chi_b(G)/b \leq (b \cdot w(f_b))/b \leq \chi^*(G) + \epsilon$. Further, this works for any $b > 2n/\epsilon$, so we have our desired result. \[\square\]

We next address the relation between $\omega_f$ and $\chi_f$ for infinite graphs with the following.

**Theorem 2.2** If $G$ is an infinite graph, then $\omega_f(G) = \overline{\chi_f}(G)$.

**Proof.** Clearly $\omega_f(G) \geq \omega_f(H) = \chi_f(H)$ for any finite subgraph $H$ of $G$, so $\omega_f(G) \geq \overline{\chi_f}(G)$.

On the other hand, if $\omega_f(G) > \overline{\chi_f}(G)$, then $G$ has a fractional clique $g$ with $w(g) > \overline{\chi_f}(G)$. Since $w(g)$ is a (possibly) infinite sum, we may find a finite partial sum arbitrarily close to $w(g)$; specifically, we may find one greater than $\overline{\chi_f}(G)$. But this finite partial sum is simply a weighting of a finite subset $U$ of $V(G)$. Let $g'$ be $g$ restricted to $U$. Any independent set of $G[U]$ (the finite subgraph induced by $U$) must also be an independent
set of $G$, and so $g'$ is a fractional clique of $G[U]$. But then $\overline{\chi_f}(G) < w(g') \leq \omega_f(G[U]) = \chi_f(G[U])$, which is a contradiction. Thus $\omega_f(G) > \overline{\chi_f}(G)$ must be false, and our result follows. □

Since we have examples of infinite graphs for which $\overline{\chi_f}(G) < \chi_f(G)$ (see Leader [8] and the following section), we know that, unlike the case of finite graphs, $\omega_f$ and $\chi_f$ can differ in infinite graphs.

### 2.2 Construction of Graphs with $\overline{\chi_f}(G) < \chi_f(G) < \infty$

We first define graphs $G^n$ and $G^{n,m}$. We let $V(G^n) = L \cup R$, where $L$ is the set of positive integers $N = \{1, 2, 3, \ldots\}$. For every size $n$ subset $N$ of $L$ we put a distinct copy of $K_n$ in $R$, and adjoin each vertex of this $K_n$ to a distinct vertex of $N$. Thus every vertex in $R$ is adjacent to exactly one vertex in $L$, and to $n - 1$ vertices in $R$. $G^{n,m}$ is defined identically, except that $L = [m] = \{1, 2, \ldots, m\}$. $G^{n,m}$ may alternately described by starting with a complete $n$-regular hypergraph on $m$ vertices, and then forming a graph by replacing each hyperedge with a new, distinct copy of $K_n$, and adjoining each vertex of this new $K_n$ to a distinct vertex from the original hyperedge.

We now define the graphs $G^n_{r,s}$ and $G^{n,m}_{r,s}$ by replacing the vertices of $G^n$ and $G^{n,m}$, respectively, with circulant graphs\(^\text{10}\); we shall refer to these circulant graphs within $G^n_{r,s}$ and $G^{n,m}_{r,s}$ as “nodes.” To be precise, we replace each vertex in $L$ with a $C(r)$, each vertex in $R$ with a $C(s)$, and all possible edges are drawn between two nodes exactly when their original vertices were adjacent in $G^n$ or $G^{n,m}$. Alternately, recalling the lexicographic product, $L$ may be thought of as $\overline{K}_{\infty}[C(r)]$ or $\overline{K}_m[C(r)]$ (in $G^n_{r,s}$ or $G^{n,m}_{r,s}$, respectively), and

\(^{10}\) Recall that $C(r)$ is a circulant graph with $\chi_f(C(r)) = r$. 

Figure 3: The graph $G_{r,s}^{2,3}$. $n = 2$ since $R$ is comprised of disjoint $K_2[C(s)]$’s. $m = 3$ is the number of nodes in $L$. The nodes shown in $L$ and $R$ represent copies of $C(r)$ and $C(s)$, respectively. The edges shown actually represent all possible edges between nodes.

If $U \subset V(G)$, we let $G[U]$ denote the subgraph of $G$ induced by $U$. From Lemma 1.5 and our observations about $L$ and $R$, we see that

$$
\chi_f(G_{r,s}^n[L]) = \chi_f(G_{r,s}^n[m][L]) = \chi_f(C_r) = r
$$

and similarly,

$$
\chi_b(G_{r,s}^n[L]) = \chi_b(G_{r,s}^n[m][L]) = \chi_b(C_r) \geq br
$$

A little thought reveals that every finite subgraph of $G_{r,s}^n$ must also be a subgraph of some $G_{r,s}^n$, so $\chi_f(G_{r,s}^n) = \lim_{m \to \infty} \chi_f(G_{r,s}^n[m])$ (since $H \subset G$ implies $\chi_f(H) \leq \chi_f(G)$).

We are now ready to start our calculations.

**Theorem 2.3** $\chi_f(G_{r,s}^n) = r + ns$.
Proof. Since $\chi_f(G^n_r[L]) = r$ and $\chi_f(G^n_r[R]) = ns$, $\chi_f(G^n_r) \leq r + ns$ is immediate.

To prove equality, it suffices to show that $\chi_b(G^n_r) \geq (r + ns)b$ for all $b \in \mathbb{N}$. This, in turn, is true if, for all $b \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\chi_b(G^{m,m}_{r,s}) \geq (r + ns)b$ (since $\chi_b(G^n_r) \geq \chi_b(G^{m,m}_{r,s})$ for all $b$ and $m$).

Fix $b \in \mathbb{N}$. Whatever $\chi_b(G^n_r)$ is (say, $c$), any optimal $b$-fold coloring of any $G^{n,m}_{r,s}$ will use no more than $c$ colors. Suppose that $C(r)$ has $a$ vertices, and consider the set of colors put on any copy of $C(r)$ in $L$. We are assigning $b$ colors to each vertex in that node, but more to the point, we are assigning no more than $ab$ of a finite set of $c$ colors to this node. There are only a finite number of ways to do this, so if we color enough nodes, we will necessarily have nodes with identical color sets. What’s more, with $a$, $b$, and $c$ fixed, if we take $m$ (the number of nodes in $L$) large enough, we can guarantee that at least $n$ nodes in $L$ will share identical color sets in any optimal $b$-fold coloring of $V(G^{m,m}_{r,s})$ (this observation follows from a simple pigeon-hole argument). In such a $b$-fold coloring of such a $G^{n,m}_{r,s}$, consider the $K_n[C(s)]$ in $R$ which corresponds to these $n$ identically colored nodes in $L$. Since every vertex of every node of this $K_n[C(s)]$ is adjacent to these same colors, a completely disjoint color set must be used to color these vertices of $R$. So at best, we can color the nodes of $L$ using $\chi_b(C(r)) \geq rb$ colors, and some copy of $K_n[C(s)]$ with $\chi_b(K_n[C(s)]) \geq \text{nsb different}$ colors. Thus $\chi_b(G^{n,m}_{r,s}) \geq (r + ns)b$, as desired. □

We next turn our attention to the problem of finding $\bar{\chi}_f(G^n_{r,s})$, which in turn requires computing $\chi_f(G^{n,m}_{r,s})$. This can’t be done exactly in most cases, but can be expressed in terms of a root of a simple polynomial. For convenience, we define $Q_f$ to be the set of all rationals in $\{1\} \cup [2, \infty)$, that is, the set of all fractional chromatic numbers of finite graphs (see [11]).
Theorem 2.4  For any $n \in \mathbb{N}$ and $r, s \in Q_f$, let $p_0$ be the real root of $rx^n + nsx - r = 0$ in $(0, 1)$. Then $\chi_f(G^n r, p_0)$.

Proof. Fix $n \in \mathbb{N}$ and $r, s \in Q_f$. We first observe that $f(x) = rx^n + nsx - r$ has a single real root in $(0, 1)$, since $f'(x) = rnx^{n-1} + ns > 0$ for $x \geq 0$, $f(0) = -r$ and $f(1) = ns$.

Since $\chi_f(G^n_{r,s}) = \lim_{m \to \infty} \chi_f(G^n_{r,s,m})$, it suffices to prove the following two inequalities:

(I) $\chi_f(G^n_{r,s,m}) \leq r/p_0$ for all $m \in \mathbb{N}$
(II) $\lim_{m \to \infty} \omega_f(G^n_{r,s,m}) \geq r/p_0$

since $\omega_f(G) = \chi_f(G)$ for any finite $G$. 
(1) $\chi_f(G_{r,s}^n) \leq r/p_0$

Fix $m \in \mathbb{N}$, and consider the maximal independent sets of $G_{r,s}^n$. We may fully describe any such set with a single parameter $p$ (up to a few irrelevant decisions\(^\text{11}\)). Denote such an independent set by $I_p$, where $p$ is the fraction of nodes in $L$ which have at least one vertex in $I_p$. We say that such a node is *covered* by $I_p$. Putting even a single vertex from such an $L$ node in $I_p$ excludes from $I_p$ anything in $R$ adjacent to that node. So if $I_p$ is to be maximal, from every covered node in $L$ we must put in $I_p$ a maximal independent set from that node (which is a copy of $C_{(r)}$). Since all maximal independent sets of $C_{(r)}$ are the same up to isomorphism, our choice of this set is irrelevant. Next consider any of one of the disjoint copies of $K_n[C_{(s)}]$ in $R$. Each node in this subgraph has a “matched” node in $L$, and we may only include in $I_p$ vertices from a node of $R$ if its matching node in $L$ is not covered by $I_p$ (every vertex in a node of $R$ is adjacent to every vertex in its matching node in $L$). So in choosing vertices from this $K_n[C_{(s)}]$ for $I_p$, we need only consider nodes whose matching nodes in $L$ are not covered. Further, once we include in $I_p$ a vertex from any one node of this $K_n[C_{(s)}]$, we exclude all its other nodes from $I_p$, since this vertex is adjacent to every vertex in every other node of this $K_n[C_{(s)}]$. So $I_p$ intersects at most one node of any $K_n[C_{(s)}]$. The choice of which “match-uncovered” node is irrelevant for our purposes. Once we have selected the node, since we want $I_p$ to be maximal, we must take a maximal independent set from that copy of $C_{(s)}$. Again, which one is irrelevant up to isomorphism. Thus, by specifying only $p$, we have (excepting a few equivalent choices) fully described what the maximal independent sets $I_p$ must be.

\(^{11}\)We would like to say “up to isomorphism”, but this is not strictly true. However, it behaves this way for our purposes.
Since these are our only maximal independent sets, they will be the only independent sets to receive positive weight in our optimal fractional coloring. In particular, we wish to limit ourselves to weighting sets with the “best” value of $p$. This value will be the one for which all such $I_p$ cumulatively place the same total weight on vertices in $L$ and $R$. To this end, let $p$ be fixed, and imagine $I_p$ to be a random variable; specifically, an independent set chosen uniformly and at random from the finite number of maximal independent sets with parameter $p$. We then wish to equate $\Pr\{v \in I_p \mid v \in L\}$ and $\Pr\{v \in I_p \mid v \in R\}$. Note that if we choose a maximal independent set uniformly and at random in $C(r) = C_{ab}$, the probability that a given vertex is in this set is exactly $b/a = 1/r$. Since $p$ is the fraction of nodes in $L$ which are covered by $I_p$,

\[
\Pr\{v \in I_p \mid v \in L\} = \Pr\{v \text{ is in a node covered by } I_p\}.
\]

\[
\Pr\{v \text{ is in an independent set of its node}\} = \frac{p}{r}.
\]

Given $v \in R$, for the event $v \in I_p$ to occur, three conditions must be met: (i) the matching node in $L$ of $v$‘s node must not be in $I_p$, (ii) $v$‘s node must be chosen from among all such “unmatched” nodes in its copy of $K_n[C_{(v)}]$, and (iii) $v$ must be in an independent set chosen from its node. Now, $\Pr\{(i)\} = 1 - p$, and $\Pr\{(iii)\} = 1/s$, but $\Pr\{(ii)\}$ is conditional on the number of other unmatched nodes in this copy of $K_n[C_{(v)}]$. We will let the index $k$ count the total number of such unmatched nodes (including $v$‘s), and $K$ will be $k - 1$ (the number of other unmatched nodes). $\Pr\{K = k - 1\}$ for any fixed value of $k$ is described as follows: if we have a huge (size $m$) pool of objects, from which a fraction $p$ are being chosen (”matched”), and we consider a specific collection of
$n-1$ of these objects (before choosing), what is the probability that exactly $n-k$ of these will be chosen (so that $k-1$ are “unmatched”)? The answer to this is not easy; because we are choosing from a finite sample, if one of our specified objects is chosen, it affects the probability that others are chosen. However, when $m >> n$, this affects is negligible, and we can approximate this process by letting each of our $n-1$ specified objects be chosen independently with probability $p$. That is, as $m$ gets large we may approximate this probability by a Bin$(n-1, p)$ distribution, and we get

$$
\text{Pr}\{(ii)\} = \sum_{k=1}^{n} \text{Pr}\{(ii) \mid K = k - 1\} \cdot \text{Pr}\{K = k - 1\}
$$

and

$$
\text{Pr}\{v \in I_p \mid v \in R\} = \text{Pr}\{(i)\} \cdot \text{Pr}\{(ii)\} \cdot \text{Pr}\{(iii)\}
$$

$$
\approx \frac{1}{ns} \sum_{k=1}^{n} \binom{n}{k} (1-p)^{k-1} p^{n-k}
$$

$$
\approx \frac{1}{ns} (1 - p^n).
$$

Now we set $\text{Pr}\{v \in I_p \mid v \in L\} = \text{Pr}\{v \in I_p \mid v \in R\}$, and get $rp^n + nsp - r = 0$. If $p_0$ is the real root of this polynomial in $(0, 1)$, then both of the above probabilities are equal to $p_0/r$. So each $v \in G_{r,s}^{m,m}$ occurs in exactly a fraction $p_0/r$ of the maximal independent sets with parameter $p_0$. If we distribute total weight $r/p_0$ equally among all such independent sets, then each vertex in $G_{r,s}^{m,m}$ will be in independent sets of total weight exactly $(r/p_0)(p_0/r) = 1$. Thus we have created a valid fractional coloring with total weight $r/p_0$. 
Of course, $p_0$ is liable to be irrational, and in any case not a multiple of $1/m$, so we can’t actually choose exactly a fraction $p_0$ of the nodes in $L$ to be covered by an $I_p$. However, as $m \to \infty$, we may choose $p$ arbitrarily close to $p_0$. Also, recall that the value of $\Pr\{ K = k - 1 \}$ was only approximated by a binomial distribution. But again, this becomes arbitrarily close to correct as $m \to \infty$, and so $r/p_0$ will be an upper bound on $\lim_{m \to \infty} \chi_f(G_{r,s}^{n,m})$. Since $G_{r,s}^{n,m} \subseteq G_{r,s}^{n,m+1}$, we know $\chi_f(G_{r,s}^{n,m})$ must be a non-decreasing function of $m$. Thus $r/p_0$ must actually be an upper bound on $\sup_{m \in \mathbb{N}} \chi_f(G_{r,s}^{n,m})$, and we’re done.

\textbf{(II)} $\lim_{m \to \infty} \omega_f(G_{r,s}^{n,m}) \geq r/p_0$

Given $G_{r,s}^{n,m}$ as before, we wish to produce a fractional clique $g$ with weight $r/p_0$ (more correctly, a sequence of fractional cliques with weights which will approach $r/p_0$ as $m \to \infty$). We assign total weight $\alpha$ to $L$, divided evenly among these vertices (each vertex gets weight $\alpha/|L|$), and weight $\beta$ to $R$, also evenly distributed, so that $w(g) = \alpha + \beta$. We require that $g$ put weight at most 1 in any independent set. Since the $I_p$’s are the only maximal independent sets in $G$, we need only worry about the weight on them. The total weight put on any $I_p$ is

$$w_{\alpha,\beta}(p) = \alpha \cdot \frac{|I_p \cap L|}{|L|} + \beta \cdot \frac{|I_p \cap R|}{|R|}.$$ 

But $\frac{|I_p \cap L|}{|L|}$ is just the previously calculated $\Pr\{ v \in I_p \mid v \in L \}$, and similarly for $R$, so

$$w_{\alpha,\beta}(p) = \frac{\alpha p}{r} + \frac{\beta (1 - p^n)}{ns}.$$ 

Now, let us set

$$\alpha = \frac{r}{p_0} \left( \frac{r p_0^{n-1}}{s + r p_0^{n-1}} \right), \quad \beta = \frac{r}{p_0} \left( \frac{s}{s + r p_0^{n-1}} \right).$$
where $p_0$ is the real root of \( rp^n + ns - r = 0 \) in \((0, 1)\). This selection of $\alpha$ and $\beta$ gives us the following:

(i) $\alpha + \beta = r/p_0$.

(ii) Taking derivatives of $w_{\alpha, \beta}(p)$ with respect to $p$ gives

\[
 w'_{\alpha, \beta}(p) = \frac{\alpha}{r} - \frac{\beta}{s}p^{n-1}, \quad w'_{\alpha, \beta}(p_0) = 0; \quad w''_{\alpha, \beta}(p) = -\frac{\beta(n-1)}{s}p^{n-2} \leq 0 \text{ for } p \geq 0.
\]

From the above, we see that $w_{\alpha, \beta}(p)$ attains its maximum value on $p \in [0, 1]$ at $p_0$.

(iii) Since $(1 - p_0^m)/ns = p_0/r$, we have $w_{\alpha, \beta}(p_0) = (p_0/r)(\alpha + \beta) = 1$.

This shows that 1 is the largest value given by $g$ to any $I_p$, and thus to any independent set of $G$. Thus $g$ is a valid fractional clique, and has total weight $r/p_0$.

Again, we must be careful. As noted at the end of part(I), $\Pr\{v \in I_p \mid v \in R\}$ is not quite $(1 - p^n)/ns$. However, it will approach this value as $m \to \infty$. So while no $G_{r, s}^{n,m}$ will actually have fractional clique number equal to $r/p_0$, as $m$ gets large we may construct fractional cliques of $G_{r, s}^{n,m}$ with values arbitrarily close to $r/p_0$. Since fractional clique number is a maximization LP, these values provide a lower bound on $\omega_f$, so $\lim_{m \to \infty} \omega_f(G_{r, s}^{n,m}) \geq r/p_0$ as desired. \(\Box\)

### 2.3 The Behavior of $\chi_f(G)$ vs. $\overline{\chi_f}(G)$

We have established the existence of graphs for which $\overline{\chi_f}(G) < \chi_f(G) < \infty$. Next we might well ask, “For which $x < y < \infty$ does there exist a graph with $x = \overline{\chi_f}(G)$ and $y = \chi_f(G)$?” We are only concerned with such $(x, y)$ pairs where $2 < x < y$, since $\chi_f(G) = 2$ implies $G$ is bipartite, even for infinite graphs.
We define

\[ S_n = \{ (x, y) \in \mathbb{R}^2 : x = \chi_f(G_{r,s}^n), \ y = \chi_f(G_{r,s}^n) \text{ for some } r, s \in \mathbb{Q}_f \} = \{ (x, y) \in \mathbb{R}^2 : \exists r, s \in \mathbb{Q}_f \text{ s.t. } \frac{r}{x} \in (0, 1), \left( \frac{r}{x} \right)^n + \frac{ns}{x} - 1 = 0, \ y = r + ns \} . \]

Note that each point in \( S_n \) is generated by an ordered pair \((r, s) \in \mathbb{Q}_f^2\). Because the definition of \( S_n \) generally involves a high order polynomial, it is difficult to describe this set more precisely. \( S_2 \), however, only involves a quadratic, and we may solve for \( x \) in terms of \( r \) and \( s \):

\[ x = \frac{r^2}{-s + \sqrt{r^2 + s^2}} . \]

Setting \((r, s) = (1, 1)\) generates \((x, y) = (1 + \sqrt{2}, 3)\), which is an isolated point of \( S_2 \). Holding one of \( r \) or \( s \) fixed at 1 and letting the other one increase from 2 generates a curve which quickly approaches the line \( y = x + 1 \). Finally, \( \{(r, s) : r, s \geq 2\} \) generates a solid\(^{12}\), roughly cone-shaped region with point at \((x, y) = (2 + 2\sqrt{2}, 6)\) (see Figure 4).

For higher values of \( n \), we may use root-finding software to plot \( S_n \), and we see a set of the same general shape as \( S_2 \). Also, we may bound the ratio \( x/y \) for \( S_2 \) and for all \( S_n \).

**Theorem 2.5** \[ \frac{\chi_f(G_{r,s}^2)}{\chi_f(G_{r,s}^2)} \leq \frac{5}{4} \] This bound is tight.

**Proof.** We have just seen that

\[ \chi_f(G_{r,s}^2) = x = \frac{r^2}{-s + \sqrt{r^2 + s^2}} . \]

\(^{12}\)The region as described is not actually solid, as it is generated only by rational \( r \) and \( s \). However, if our goal is to cover as much of the plane as possible, we may construct \( G \) to be a disjoint sequence of \( G_{r,s}^n \) in which the \( r \)'s and \( s \)'s approach any desired real limits. It is easy to show that \( \chi_f(G) \) and \( \chi_f(G) \) actually take on their expected values as indicated by these limits.
If we solve for $s$, we get $s = \frac{1}{2} \left( x - \frac{r^2}{x} \right)$. We now fix $x$, and see that

$$
\chi_f(G^2_{r,s}) = y = r + 2s = r + x - \frac{r^2}{x}
$$

$$
\frac{dy}{dr} = 1 - \frac{2r}{x} = 0 \quad \text{at} \quad r = x/2
$$

$$
\frac{d^2y}{dr^2} = -2/x < 0
$$

If we take $x = \overline{\chi}_f$ to be a fixed value, we may still vary the value of $y = \chi_f$ by varying $r$ and $s$. And the above shows that, as a function of $r$, $y$ is maximized at $r = x/2$. At this value, $y = \frac{5}{4}x$, that is, $\chi_f(G^2_{r,s}) = \frac{5}{4}\overline{\chi}_f(G^2_{r,s})$. This is the largest $\chi_f$ can be relative to $\overline{\chi}_f$, and our construction guarantees that this ratio is actually achieved. □

**Theorem 2.6** \( \frac{\chi_f(G^2_{r,s})}{\overline{\chi}_f(G^2_{r,s})} \leq 2 \) for any integer $n \geq 2$ and $r, s \in Q_f$. Further, if we keep $r = ns$, then this bound is tight as $n \to \infty$.

**Proof.** Since any $G^{n,m}_{r,s}$ contains $C(r)$ and $K_n[C(s)]$ as subgraphs, we must have $\chi_f(G^{n,m}_{r,s}) \geq \max\{r, ns\}$, so

$$
\chi_f(G^n_{r,s}) = r + ns \leq 2\max\{r, ns\} \leq 2\chi_f(G^{n,m}_{r,s}),
$$
which proves the first half of our claim.

Next, set \( r = ns \), so that \( p_0 \) is the root of \( x^n + x - 1 = 0 \) in \((0, 1)\). Let \( n \to \infty \). Then \( p_0 \to 1 \) since \( x^n \to 0 \) for any \( x \) in \((0, 1)\). So we have \( \chi_f(G_{r,s}^n) = r/p_0 \to r \) and \( \chi_f(G_{r,s}^n) = r + ns = 2r \). \( \square \)

We may now define \( S = \cup_{n=2}^{\infty} S_n \) to be the region of the plane covered by ordered pairs of the form \((\chi_f(G_{r,s}^n), \chi_f(G_{r,s}^n))\). Again, if our goal is to cover as much of the plane as possible, we may resort to one more trick: recall from Lemma 1.5 that \( \chi_f(G[H]) = \chi_f(G)\chi_f(H) \), and note that \( \chi_f(G[H]) = \chi_f(G)\chi_f(H) \) follows immediately from this. Since any two points in \( S \) represent two known graphs, we may take their lexicographic product to get a graph with \((\chi_f(G), \chi_f(G))\) equal to the component-wise product of the two points in \( S \). More generally, by taking multiple graph products, we may now cover the following region of the plane:

\[
S' = \left\{ \left( \prod_{i=1}^{k} x_i, \prod_{i=1}^{k} y_i \right) \in \mathbb{R}^2 : (x_i, y_i) \in S \text{ for } i = 1, \ldots, k, \ k \in \mathbb{N} \right\}.
\]

In particular, for \((x, y) \in S'\), the ratio \( y/x \) is no longer bounded, since \((x^k, y^k) \in S'\) for any integer \( k \) and \((x, y) \in S\). However, attaining large ratios also requires large values of \( x \) and \( y \). So, for instance, while \((n, r, s) = (2, 4, 3)\) gives the point \((8, 10) \in S_2\), and Leader [8] constructs a graph with \((\chi_f(G), \chi_f(G)) = (8, \infty)\), it is unknown whether or not a graph exists with \((\chi_f(G), \chi_f(G)) = (8, 80)\) (for instance). So while much of the plane has been covered herein, we still have the open problem “Given any real \( x \) and \( y \) with \( y > x > 2 \), does there exist \( G \) with \( \chi_f(G) = x \) and \( \chi_f(G) = y \)?”