3 Fractional Ramsey Numbers

Since the definition of Ramsey numbers makes use of the clique number of graphs, we may define fractional Ramsey numbers simply by substituting fractional clique number into this definition. Recall that the Ramsey arrow notation “$n \rightarrow (k, l)$” means that “For any red/blue-coloring of the edges of $K_n$, there must a either a red $k$-clique or a blue $l$-clique.” More precisely, whenever $K_n = H_1 \oplus H_2$, then we must have $\omega(H_1) \geq k$ or $\omega(H_2) \geq l$. This edge-decomposition of $K_n$ may be thought of as an edge 2-coloring. The Ramsey number $r(k, l)$ is the least positive integer $n$ for which this statement is true. Very few exact values for $r(k, l)$ are known. For instance, $r(5, 5)$ is only known to be somewhere between 43 and 55. And while the growth rate of $r(k, k)$ is known to be an exponential function of $k$, the best known lower and upper bounds on this growth are (roughly) $\sqrt{2}^k$ and $4^k$, respectively.

Now, we may define the fractional Ramsey arrow notation $n \rightarrow_f (x, y)$ to mean that, whenever $K_n = H_1 \oplus H_2$, we must have $\omega_f(H_1) \geq x$ or $\omega_f(H_2) \geq y$. Then the fractional Ramsey number $r_f(x, y)$ is the least positive integer $n$ for which $n \rightarrow_f (x, y)$. Note that we may meaningfully take $x$ and $y$ to be any real numbers greater than 1.

All graphs in this chapter are implicitly assumed to be finite. When we speak of $K_n = H_1 \oplus H_2$ as an edge 2-coloring, it is implicitly assumed that we are invoking the condition on $\oplus$ that $E(H_1) \cap E(H_2) = \emptyset$. However, a little thought reveals that the above Ramsey definitions with or without this condition are equivalent. We only note this because, in forthcoming constructions, it is sometimes convenient to take $H_i$’s which are not edge-disjoint. However, if we find such $H_i$’s with $\omega_f(H_i) < x_i$, removing edges from some $H_i$’s
to make them edge-disjoint does not cause this condition to be violated. So we adhere to the convention of \( E(H_1) \cap E(H_2) = \emptyset \) only as is convenient.

### 3.1 The Value of \( r_f(x, y) \)

Unlike the ordinary Ramsey numbers, the exact value of \( r_f(x, y) \) is known for any \( x, y \geq 2 \).

**Theorem 3.1** Let \( x, y \in \mathbb{R} \) with \( x, y > 1 \). Express \( x \) and \( y \) as \( x = k + \varepsilon \) and \( y = l + \delta \), where \( k, l \in \mathbb{N} \) and \( 0 < \varepsilon, \delta \leq 1 \). Let \( q = \min\{\lceil \varepsilon l \rceil, \lceil \delta k \rceil \} \). Then \( r_f(x, y) = kl + q \).

**Proof.** We first establish two basic facts about \( \varepsilon, \delta, k, l \) and \( q \) as given above.

(i) Either \( q = \lceil \varepsilon l \rceil \geq \varepsilon l \) or \( q = \lceil \delta k \rceil \geq \delta k \), so that either \( q/l \geq \varepsilon \) or \( q/k \geq \delta \).

(ii) \( q \leq \lceil \varepsilon l \rceil < \varepsilon l + 1 \) and \( q \leq \lceil \delta k \rceil < \delta k + 1 \), so that \( (q - 1)/l < \varepsilon \) and \( (q - 1)/k < \delta \).

Now, \( r_f(x, y) \) is clearly symmetric in \( x \) and \( y \), and \( r_f(x, y) = \lceil x \rceil \) is easily checked for \( y \in (1, 2] \), so we restrict our attention to the case of \( x \) and \( y \) both greater than 2. Otherwise, take \( x, y, k, \varepsilon, l, \delta \) and \( q \) as given above, and set \( n = kl + q \). We establish that \( r_f(x, y) = n \) in two steps: first, show that \( n \nrightarrow (x, y) \); then, construct a decomposition \( K_{n-1} = H_1 \oplus H_2 \) with \( \omega_f(H_1) < x \) and \( \omega_f(H_2) < y \) (which shows that \( n - 1 \nrightarrow (x, y) \) is false).

To show \( n \nrightarrow (x, y) \), let \( K_n = H_1 \oplus H_2 \) be any edge 2-coloring of \( K_n \) (so that \( E(H_1) \cap E(H_2) = \emptyset, \omega(H_1) = \alpha(H_2) \) and vice versa). If \( \omega(H_1) \geq k + 1 \), then \( \omega_f(H_1) \geq k + 1 \geq x \), and we’re done. Since \( \omega(H_2) \geq l + 1 \) similarly implies \( \omega_f(H_2) \geq y \), we may suppose that \( \alpha(H_2) = \omega(H_1) \leq k \) and \( \alpha(H_1) = \omega(H_2) \leq l \). Then by Lemma 1.3 and (i)
above, either $q/l \geq \varepsilon$, in which case
\[
\omega_f(H_1) \geq \frac{n}{\alpha(H_1)} \geq \frac{kl + q}{l} = k + \frac{q}{l} \geq k + \varepsilon,
\]
or $q/k \geq \delta$, which gives
\[
\omega_f(H_2) \geq \frac{n}{\alpha(H_2)} \geq \frac{kl + q}{k} = l + \frac{q}{k} \geq l + \delta.
\]
That one of these holds is exactly the statement $n \not\rightarrow (x, y)$.

To achieve $K_{n+1} = H_1 \oplus H_2$ with $\omega_f(H_1) < x$ and $\omega_f(H_2) < y$, we take $H_1 = C_{(n-1),l}$ and $H_2 = \overline{H_1}$. By Lemma 1.4 and (ii) above, this immediately gives
\[
\omega_f(H_1) = \frac{n - 1}{l} = \frac{kl + q - 1}{l} = k + \frac{q - 1}{l} < k + \varepsilon = x
\]
and
\[
\omega_f(H_2) = \frac{n - 1}{[(n - 1)/l]}.
\]
We note that $(n - 1)/l = k + \frac{q - 1}{l} < k + \varepsilon$, and also that $(n - 1)/l \geq k$ since $q$ is at least 1. So $[(n - 1)/l] = k$, and applying (ii) again gives
\[
\omega_f(H_1) = \frac{n - 1}{k} = \frac{kl + q - 1}{k} = l + \frac{q - 1}{k} < l + \delta = y.
\]
These are exactly the properties we required of $H_1$ and $H_2$, so $n - 1 \not\rightarrow (x, y)$ is false, and we’re done. \(\square\)

**Corollary 3.2** If $k \geq l \geq 2$ are integers, then $r_f(k, l) = kl - k$. \(\square\)

Note that, while $r(k, k)$ grows exponentially in $k$, $r_f(x, x)$ only grows quadratically in $x$. Two plots of $r_f(x, x)$ vs. $x$ are shown in Figure 5.
Figure 5: Two graphs of the function \( y = r_f(x, x) \). Although this is a step function, its growth roughly conforms to a parabola. Large jumps occur at integer values of \( x \), while size 1 jumps occur at even intervals between integers.

### 3.2 Multicolor Fractional Ramsey Numbers

We may extend the definition of Ramsey numbers by using more than two colors on the edges of \( K_n \). We let \( n \rightarrow (k_1, \ldots, k_p) \) mean that, whenever \( K_n = H_1 \oplus \cdots \oplus H_p \), we must have \( \omega(H_i) \geq k_i \) for some \( i \in \{1, \ldots, p\} \). The Ramsey number \( r(k_1, \ldots, k_p) \) is the least positive integer \( n \) for which this holds.

Similarly, we take \( n \rightarrow^f (x_1, \ldots, x_p) \) to mean that, whenever \( K_n = H_1 \oplus \cdots \oplus H_p \), we must have \( \omega_f(H_i) \geq k_i \) for some \( i \in \{1, \ldots, p\} \), and the fractional Ramsey number \( r(x_1, \ldots, x_p) \) is the least positive integer for which this is true.

We may derive a recursive upper bound on \( r_f \) as follows:

**Theorem 3.3** Let \( x_1, \ldots, x_p \geq 2 \). Then

\[
r_f(x_1, \ldots, x_p) \leq \left\lfloor (r_f(x_1, \ldots, x_{p-1}) - 1)x_p \right\rfloor.
\]

**Proof.** Let \( n = \left\lfloor (r_f(x_1, \ldots, x_{p-1}) - 1)x_p \right\rfloor \), and let \( K_n = H_1 \oplus \cdots \oplus H_p \) be any \( p\)-
coloring of $E(K_n)$. Let $G = H_1 \oplus \cdots \oplus H_{p-1}$. If $\omega(G) \geq r_f(x_1, \ldots, x_{p-1})$, then $G$ contains a complete subgraph large enough to guarantee that $\omega_f(H_i) \geq x_i$ for some $i \in \{1, \ldots, p - 1\}$, and we’re done. So we may suppose that $\omega(G) \leq r_f(x_1, \ldots, x_{p-1}) - 1$.

Since $\alpha(H_p) = \omega(G)$ we have

$$\omega_f(H_p) = \frac{n}{\alpha(H_p)} \geq \frac{[(r_f(x_1, \ldots, x_{p-1}) - 1)x_p]}{r_f(x_1, \ldots, x_{p-1}) - 1} \geq x_p$$

as required. \[\square\]

In fact, there are no known instances where this upper bound does not provide the correct value of $r_f$. For example, in the case of $p = 2$, the given bound reduces to

$$r_f(x, y) \leq \min \{\lceil \lfloor x \rfloor - 1 \rceil y, \lceil \lfloor y \rfloor - 1 \rceil x \}$$

which is easily shown to be the value derived for $r_f(x, y)$ in Theorem 3.1. Of course, $r_f$ is symmetric in its arguments, so in order for the expression in Theorem 3.3 to yield the best possible bound, we must at each recursive step choose the best $x_i$ value to play the role of $x_p$ in this expression. All of this implies the following conjecture.

**Conjecture 3.4** Let $x_1, \ldots, x_p \geq 2$. Then for some $i \in \{1, \ldots, p\}$,

$$r_f(x_1, \ldots, x_p) = \lceil (r_f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p) - 1)x_i \rceil \ . \ \square$$

In the case that the arguments are integers, the “correct” choice of $x_i$ is simply the largest argument, so for integers $k_p \geq k_{p-1} \geq \cdots \geq k_1 \geq 2$ we have

$$r_f(k_1, \ldots, k_p) \leq \prod_{i=1}^{p} k_i - \prod_{i=2}^{p} - \cdots - k_{p-1}k_p - k_p .$$
On the other hand, it is easily shown\(^\text{13}\) that
\[
 r_f(k_1, \ldots, k_p) > \prod_{i=1}^{p} (k_i - 1)
\]
so our upper bound is at least on the right order of magnitude.

There are several other instances where our conjecture is known to be true, the most significant being the case where the arguments are all the same integer.

**Theorem 3.5** For integers \( k, p \geq 2 \),
\[
 r_f(k, k, \ldots, k) = k^p - k^{p-1} - \cdots - k.
\]

We postpone the rather long proof of this theorem to mention a few other instances where Conjecture 3.4 is true. In the following three cases, the value of \( r_f \) is that given by Conjecture 3.4, and we give \( K_{r_f-1} = H_1 \oplus H_2 \oplus H_3 \), where each \( H_i \) is the indicated circulant graph, and \( I_i \) one of its independent sets of required size. We take \( V(K_{r_f-1}) = V(H_i) \) to be \( \{0, 1, \ldots, r_f - 2\} \), and note that \( \omega_f(H_i) = (r_f - 1)/|I_i| \) (by Lemma 1.3).

\begin{itemize}
\item \( r_f(x, y, z) = 20 \), \( \frac{19}{7} < x, y \leq 3 \), \( \frac{19}{3} < z \leq 4 \).
\[ H_1 = \langle \{2, 8\} \rangle_{19}, \quad I_1 = \{0, 3, 6, 9, 12, 15, 18\}, \quad \omega_f(H_1) = 19/7. \]
\[ H_2 = \langle \{4, 6\} \rangle_{19}, \quad I_2 = \{0, 1, 2, 3, 10, 11, 12\}, \quad \omega_f(H_2) = 19/7. \]
\[ H_3 = \langle \{1, 3, 5, 7, 9\} \rangle_{19}, \quad I_3 = \{0, 2, 4, 6, 8\}, \quad \omega_f(H_3) = 19/5. \]
\item \( r_f(x, y, z) = 28 \), \( \frac{27}{10} < x \leq 3 \), \( \frac{27}{7} < y, z \leq 4 \).
\[ H_1 = \langle \{1, 4, 6\} \rangle_{27}, \quad I_1 = \{0, 2, 5, 7, 10, 12, 15, 17, 20, 22\}. \]
\[ \omega_f(H_1) = 27/10. \]
\end{itemize}

\(^{13}\)See [6].
There is a final general case where Conjecture 3.4 is true. Let $k_1, \ldots, k_p \geq 2$ be integers. Then if each of $\varepsilon_1, \ldots, \varepsilon_p > 0$ is sufficiently small, we have

$$r_f(k_1 + \varepsilon_1, \ldots, k_p + \varepsilon_p) = 1 + \prod_{i=1}^{p} k_i.$$

That this is the value indicated by Conjecture 3.4 is easily checked. For the lower bound, consider $K_{k_1k_2} = K_{k_1}[K_{k_2}]$. Every edge from this graph comes either from $K_{k_1}$ (between two copies of $K_{k_2}$) or $K_{k_2}$ (within a copy of $K_{k_2}$). If we let $H_1$ contain all such $K_{k_1}$ edges, and $H_2$ all such $K_{k_2}$ edges, then $H_1 = K_{k_1}[K_{k_2}]$ and $H_2 = K_{k_1}[K_{k_2}]$. By Lemma 1.5, $\omega_f(H_1) = k_1$ and $\omega_f(H_2) = k_2$. More generally, if we take successive lexicographic products of the $K_{k_i}$’s, we get a complete graph on $k_1k_2\cdots k_p$ vertices. And if we let $H_i$ consist of all edges which come from $K_{k_i}$, then

$$H_i = \overline{K_{k_1\cdots k_{i-1}}K_{k_i}[K_{k_{i+1}\cdots k_p}]}.$$

$$K_{k_1k_2\cdots k_p} = H_1 \oplus \cdots \oplus H_p,$$ and
\[ \omega_f(H_i) = k_i < k_i + \varepsilon_i. \]

This shows that \( k_1 \cdots k_p \rightarrow f(k_1 + \varepsilon_1, \ldots, k_p + \varepsilon_p) \) is false.

Finally note that, while choosing the largest \( x_i \) gives the best bound in the case that all arguments are integers, this is not in general the case. To calculate \( r_f(3.1, 3.1, 4.9) \), first note that \( r_f(3.1, 3.1) = 10 \) and \( r_f(3.1, 4.9) = 13 \). Then

\[
\begin{align*}
    r_f(3.1, 3.1, 4.9) &\leq \left[ (r_f(3.1, 3.1) - 1)4.9 \right] = 45, \quad \text{but} \\
    r_f(3.1, 3.1, 4.9) &\leq \left[ (r_f(3.1, 4.9) - 1)3.1 \right] = 38.
\end{align*}
\]

The second bound is clearly better, even though the largest \( x_i \) was not used as the recursion point.

We now return to...

**Proof of Theorem 3.5.** We restrict our attention to the case of \( k \geq 3 \), as the \( k = 2 \) case is trivially true. And since we’ve already proved Theorem 3.3, all that remains to be shown is that \( r_f(k, \ldots, k) > k^p - k^{p-1} - \cdots - k - 1 \). As in the 2-color proof, this is accomplished by constructing \( K_{r_f(k, \ldots, k)-1} = H_1 \oplus \cdots \oplus H_p \) (an edge \( p \)-coloring of the complete graph) such that \( \omega_f(H_i) < k \) for all \( i \). And as before, this is accomplished via the use of circulant graphs. We use the more general form \( \langle S \rangle_n \), and will generally suppress the \( n \) subscript, as its value will be clear in context.

In the following, we will hold \( k \) fixed and let \( p \) vary. Let \( n_p = k^p - k^{p-1} - \cdots - k - 1 \), the order of the complete graph whose edges we are \( p \)-coloring. Notice that \( n_{p+1} = kn_p - 1 \), and since \( n_1 = k - 1 \), we set \( n_0 = 1 \) for consistency. Since the \( H_i \)'s will be chosen to be circulant graphs, they will be vertex transitive, and thus \( \omega_f(H_i) = \frac{np}{\alpha(H_i)} \) by Lemma 1.3. Since we want \( \omega_f(H_i) < k \), we let \( \alpha_p \) be the desired value of \( \alpha(H_i) \) in the \( p \)-color case.
Specifically,

\[
\alpha_p = \min\{\alpha \in \mathbb{N} : \frac{n_p}{\alpha} < k\}
\]
\[
= \min\{\alpha \in \mathbb{N} : \alpha > \frac{kn_{p-1} - 1}{k} = n_{p-1} - \frac{1}{k}\}
\]
\[
= n_{p-1},
\]

with \(\alpha_1 := 1\). (Henceforth, \(n_{p-1}\) and \(\alpha_p\) are used interchangeably).

Let \(d_p = \lfloor \frac{n_p}{2} \rfloor\), so that \(K_{n_p} = \langle \{1, \ldots, d_p\} \rangle_{n_p}\). Then in order to have \(K_{n_p} = H_1 \oplus \cdots \oplus H_p\) where each \(H_i\) is a circulant graph, we require that the union of the connection distance sets of all the \(H_i\)’s be exactly \(\{1, \ldots, d_p\}\). Note that in the following construction, the \(H_i\)’s will not be edge-disjoint (the connection distance sets will not be disjoint). Let us superscript each \(H_i\) with its corresponding \(p\), and then let \(S_i^p\) be the set of connection distances defining \(H_i^p\), i.e., \(H_i^p = \langle S_i^p \rangle\). For a set of integers \(S\) and an integer \(r\), we define \(S + r := \{s + r : s \in S\}\). Similarly \(r - S := \{r - s : s \in S\}\).

We begin our construction by defining

\[
S_i := \{n_{i-1}, \ldots, n_i - n_{i-1}\}
\]
\[
h_i := (k - 1)n_i - 1 = n_{i+1} - n_i = k^{i+1} - 2k^i
\]

and then

\[
S_i^p := \{s \leq d_p : s \mod h_i \in S_i\}
\]
\[
\subset S_i \cup (S_i + h_i) \cup (S_i + 2h_i) \cup \cdots
\]

Notice that, since \(k \geq 3\), for \(p \geq 1\) we have that \(kn_{p-1} - 1 \geq 2n_{p-1}\). Then \(n_p/2 \geq n_{p-1}\), and we have

\[
n_p - n_{p-1} \geq \frac{n_p}{2}
\]
\[ \geq \left\lfloor \frac{n_p}{2} \right\rfloor = d_p \]

\[ \geq n_{p-1} \cdot \]

Therefore \( \{n_{p-1}\} \subseteq \{n_{p-1}, \ldots, d_p\} \subseteq \{n_{p-1}, \ldots, n_p - n_{p-1}\} \), and \( S^p_p = \{n_{p-1}, \ldots, d_p\} \).

And since \( n_p - n_{p-1} \leq n_p = n_{p-1} + h_{p-1} \), it follows that \( S^p_{p-1} = S_{p-1} \). Finally, note that \( S^p _i \subseteq S^{p+1}_i \).

**Example**

To help illustrate the proof, we consider the case \( k = 4 \). Here we present the items we have defined thus far for \( p = 1, 2, 3, 4 \).

\[
\begin{align*}
n_1 &= 3 \quad n_2 = 11 \quad n_3 = 43 \quad n_4 = 171 \\
d_1 &= 1 \quad d_2 = 5 \quad d_3 = 21 \quad d_4 = 85 \\
h_1 &= 8 \quad h_2 = 32 \quad h_3 = 128 \\
\alpha_1 &= 1 \quad \alpha_2 = 3 \quad \alpha_3 = 11 \quad \alpha_4 = 43 \\
S_1 &= \{1, 2\} \\
S_2 &= \{3, 4, 5, 6, 7, 8\} \\
S_3 &= \{11, 12, \ldots, 32\} \\
S_4 &= \{43, 44, \ldots, 128\} \\
S^1_i &= \{1\} \\
S^2_i &= \{1, 2\} \\
S^3_i &= \{3, 4, 5\} \\
S^4_i &= \{1, 2, 9, 10, 17, 18\} \\
S^5_i &= \{3, 4, 5, 6, 7, 8\} \\
S^6_i &= \{11, 12, \ldots, 21\} \\
S^7_i &= \{1, 2, 9, 10, 17, 18, 25, 26, 33, 34, 41, 42, \\
& \quad 49, 50, 57, 58, 65, 66, 73, 74, 81, 82\} \\
S^8_i &= \{3, \ldots, 8, 35, \ldots, 40, 67, \ldots, 72\} \\
S^9_i &= \{11, 12, \ldots, 32\} \\
S^{10}_i &= \{43, 44, \ldots, 85\}
\end{align*}
\]

We return to the example of \( k = 4 \) several times to further clarify our constructions.
We now use induction to verify that this construction actually does cover all connection distances $1, \ldots, d_p$, (i.e., that $S_1^p \cup \cdots \cup S_p^p = \{1, \ldots, d_p\}$, and therefore $K_{n_p} = H^p_1 \oplus \cdots \oplus H^p_p$). We have $S_1^1 = \{1, \ldots, d_p\}$ as a base case, so we now suppose that $S_1^p \cup \cdots \cup S_p^p = \{1, \ldots, d_p\}$. Since $S_{p+1}^p = \{n_p, \ldots, d_{p+1}\}$, we need only show that $\{1, \ldots, n_p - 1\} \subset (S_{p+1}^1 \cup \cdots \cup S_{p+1}^{p+1})$.

Take $i < p$. Since $h_i = k^{i+1} - 2k^i$, we compute that

$$n_p = k^p - k^{p-1} - k^{p-2} - \cdots - k - 1$$
$$= (k^{i+1} - 2k^i)(k^{p-i-1} + \cdots + 1) + k^i - k^{i-1} - \cdots - 1$$
$$= h_i(k^{p-i-1} + \cdots + 1) + n_i,$$

so $n_p \mod h_i = n_i$. Next, if $x \in S_1^p$, then $n_p - x \in n_p - S_i^p$, and

$$(n_p - x) \mod h_i \in [(n_p - S_i^p) \mod h_i] \subseteq [(n_p - S_i) \mod h_i]$$
$$= (n_i - S_i) \mod h_i$$
$$= (n_i - \{n_{i-1}, \ldots, n_i - n_{i-1}\}) \mod h_i$$
$$= \{n_{i-1}, \ldots, n_i - n_{i-1}\}$$
$$= S_i.$$

We also know that $n_p - x < n_p \leq d_{p+1}$, so $n_p - x \in S_{p+1}^p$. Since $S_1^p, \ldots, S_p^p$ cover $\{1, \ldots, d_p\}$ (our induction hypothesis), it then follows that $S_{p+1}^p, \ldots, S_{p+1}^p$ cover $\{n_p - d_p, \ldots, n_p - 1\}$ as well as $\{1, \ldots, d_p\}$ (recall that $S_i^p \subset S_{i+1}^p$). But since $d_p = \lceil \frac{n_p}{2} \rceil$, we have that $\{1, \ldots, d_p\} \cup \{n_p - d_p, \ldots, n_p - 1\} = \{1, \ldots, n_p - 1\}$, all of which is covered by $S_{p+1}^1, \ldots, S_{p+1}^p$, as desired. Thus all connection distances are covered, and it follows that $K_{n_p} = H^p_1 \oplus \cdots \oplus H^p_p$. 


We must now show that $\alpha(H_1^p) \geq \alpha_p$, so that $\omega_f(H_1^p) < k$. We accomplish this by

(i) defining a set $I_i^p \subseteq \{0, 1, \ldots, n_p - 1\}$,

(ii) showing that $I_i^p$ is an independent set in $H_i^p$, and

(iii) showing that $|I_i^p| \geq \alpha_p$.

**(i) Constructing $I_i^p$**

We first define the sets

$$B_i = \{0, 1, \ldots, \alpha_i - 1\},$$

$$C_i = \{0, 1, \ldots, \alpha_i - 2\},$$

and

$$D_i = B_i \cup (B_i + n_i) \cup \cdots \cup [B_i + (k - 3)n_i] \cup [C_i + (k - 2)n_i].$$

We then define our independent set $I_i^p$ to be

$$I_i^p = D_i \cup (D_i + h_i) \cup \cdots \cup (D_i + r_{i,p}h_i) \cup [B_i + (r_{i,p} + 1)h_i],$$

where

$$r_{i,p} = \max\{t : \alpha_i - 1 + (t + 1)h_i < n_p\}.$$

That is to say, $r_{i,p}$ is the largest possible value so that no element of $I_i^p$ exceeds $n_p$. Specifically, $r_{i,p} = k^{p-i} + k^{p-i-2} + \cdots + k$. To see this, add $n_i$ (which is much smaller than the span of a $D_i$, but larger than the span of $B_i$) to the leading 0 in the last $B_i$ in $I_i^p$. We get

$$\begin{align*}
(r_{i,p} + 1)h_i + n_i &= (k^{p-i-1} + \cdots + k + 1)(k - 1)n_i - (r_{i,p} + 1) + n_i \\
&= (k^{p-i} - 1)n_i + n_i - (r_{i,p} + 1) \\
&= (k^p - k^{p-1} - \cdots - k^{p-i}) - (k^{p-i-1} + \cdots + k + 1) \\
&= n_p,
\end{align*}$$
so the given value of $r_{i,p}$ implies $I^p_i$ is a subset of $\{0, \ldots, n_p - 1\}$. In the case of $i = p - 1$, this gives $r_{i,p} = 0$, and we just have $I^p_{p-1} = D_{p-1} \cup (B_{p-1} + h_{p-1})$. Finally, we set $I^p_p = B_p = \{0, \ldots, \alpha_p - 1\}$. This completes our construction.

\textbf{Example}

Here, we list independent sets $I^p_i$ and their components in the case of $k = 4$ and $p = 1, 2, 3, 4$. A better illustration of these patterns for $i = 2$ may be found in Figure 6.

\begin{align*}
B_1 &= \{0\} \\
C_1 &= \emptyset \\
D_1 &= B_1 \cup (B_1 + 3) \cup (C_1 + 6) = \{0, 3\} \\
B_2 &= \{0, 1, 2\} \\
C_2 &= \{0, 1\} \\
D_2 &= B_2 \cup (B_2 + 11) \cup (C_2 + 22) = \{0, 1, 2, 11, 12, 13, 22, 23\} \\
B_3 &= \{0, 1, \ldots, 10\} \\
C_3 &= \{0, 1, \ldots, 9\} \\
D_3 &= B_3 \cup (B_3 + 43) \cup (C_3 + 86) = \{0, 1, \ldots, 10, 43, 44, \ldots, 53, 86, 87, \ldots, 95\} \\
h_1 &= 8 \\
h_2 &= 32 \\
h_3 &= 128 \\
I^1_1 &= \{0\} \\
I^2_1 &= \{0, 3, 8\} \\
I^3_1 &= \{0, 1, 2\} \\
I^1_2 &= \{0, 3, 8, 11, 16, 19, 24, 27, 32, 35, 40\} \\
I^2_2 &= \{0, 1, 2, 11, 12, 13, 22, 23, 32, 33, 34\} \quad \text{See Figure 6} \\
I^3_2 &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \\
I^1_3 &= \{0, 3, 8, 11, 16, 19, 24, 27, 32, 35, 40, 43, 48, 51, 56, 59, 64, 67, 72, 75, 80, 83, 88, 91, 96, 99, 104, 107, 112, 115, 120, 123, 128, 131, 136, 139, 144, 147, 152, 155, 160, 163, \}
\end{align*}
(ii) \( I^p \) is an independent set of \( H^p \)

Given a subset \( I \) of the vertices of a circulant graph \( \langle S \rangle \), \( I \) is an independent set of \( \langle S \rangle \) if, for any two \( x, y \in I \), the distance between them along their shorter around the circle (i.e. their “connection distance”) is not in the set \( S \) of connection distances of the graph. We check that this condition is satisfied in each of three cases, dependent on the value of \( i \).
Figure 6: The desired independent sets $I$ of $G$ for $\rho = 2$, $\alpha = 3$, $n = 11$. Note that, while the vertices are drawn here in rows for clarity, they should be imagined to be laid out around the perimeter of a circle.

\[
\begin{align*}
I_2 &= \{B_2\} \\
I_3 &= \{D_2 \cup B_2 + h_2(0), D_2 \cup B_2 + h_2(1)\} \\
I_4 &= \{D_2 \cup D_2 + h_2(0), D_2 \cup D_2 + h_2(2), D_2 \cup D_2 + h_2(3), D_2 \cup D_2 + h_2(4), D_2 \cup D_2 + h_2(5)\}
\end{align*}
\]
(ii.a) $I_p$ is an independent set of $H_p$

Recall that $S_p = \{\alpha_p, \ldots, d_p\}$ and $I_p = \{0, \ldots, \alpha_p - 1\}$ (where $\alpha_p - 1 < n_p/2$). The elements of $I_p$ are therefore consecutive along the circle, and none are more than $\alpha_p - 1$ steps apart. Since the smallest connection distance is $\alpha_p$, no pair of vertices in $I_p$ can be adjacent.

(ii.b) $I_{p-1}$ is an independent set of $H_{p-1}$

Recall that $S_{p-1} = S_p = \{\alpha_{p-1}, \ldots, \alpha_p - \alpha_{p-1}\}$, and that $I_{p-1}$ contains exactly one copy of $D_{p-1}$. To show that $I_{p-1}$ is an independent set of $H_{p-1}$, we consider all possible connection distances within $I_{p-1}$, and observe that none of them is found in $S_{p-1}$.

Note that $I_{p-1}$ is just the union of translated copies of $B_{p-1}$ and $C_{p-1}$. Any pair of vertices in the same copy of $B$ or $C$ are at most $\alpha_{p-1} - 1$ steps apart. Since the smallest connection distance (element of $S_{p-1}$) is $\alpha_{p-1}$, these vertices cannot be adjacent, so we turn our attention to vertices in different copies of $B$ and $C$. The smallest possible distance between such a pair of vertices occurs between the last vertex $u$ in one block ($B$ or $C$) and the first vertex $v$ of the next block. We now prove that, in all possible cases, this smallest possible distance is larger than $\alpha_p - \alpha_{p-1}$ (the largest value in $S_{p-1}$).

- $u$ is last vertex in $B_{p-1} + jn_{p-1}$ and $v$ is the first vertex in $B_{p-1} + (j + 1)n_{p-1}$ with $0 \leq j < k - 3$.

In this case we have $u = (\alpha_{p-1} - 1) + jn_{p-1}$ and $v = 0 + (j + 1)n_{p-1}$, so the distance between them is $v - u = n_{p-1} - (\alpha_{p-1} - 1) = \alpha_p - \alpha_{p-1} + 1$, as desired.

- $u$ is the last vertex in $B_{p-1} + (k-3)n_{p-1}$ and $v$ is the first vertex in $C_{p-1} + (k-2)n_{p-1}$.

As above, the distance between $u$ and $v$ is $n_{p-1} - (\alpha_{p-1} - 1) = \alpha_p - \alpha_{p-1} + 1$. 
• \( u \) is the last vertex in \( C_{p-1} + (k-2)n_{p-1} \) and \( v \) is the first vertex in \( B_{p-1} + h_{p-1} \).

Here, \( u = (\alpha_{p-1} - 2) + (k-2)n_{p-1} \) and \( v = 0 + h_{p-1} = (k-1)n_{p-1} - 1 \). The distance between them is \( \alpha_p - \alpha_{p-1} + 1 \), as desired.

• \( u \) is the last vertex in \( B_{p-1} + h_{p-1} \) and \( v \) is the first vertex in \( B_{p-1} \).

In this case \( u = h_{p-1} + \alpha_{p-1} - 1 \) and \( v = 0 \). Their distance is \( n_p - (h_{p-1} + \alpha_{p-1} - 1) = \alpha_p - \alpha_{p-1} + 1 \), as desired.

Recapping... any two vertices in the same \( B \) or \( C \) block of \( I^p_{p-1} \) have connection distance less than \( \alpha_{p-1} \), and so less than any value in \( S^p_{p-1} = S_{p-1} \); any two vertices in different blocks of \( I^p_{p-1} \) have connection distance greater than \( \alpha_p - \alpha_{p-1} \), and so greater than any value in \( S^p_{p-1} \). (Keep these facts in mind for the following section.) Thus \( I^p_{p-1} \) is an independent set of \( H_{p-1} \).

\((\text{ii.c})\) \( I^p_i \) is an independent set of \( H^p_i \) for \( 1 \leq i < p - 1 \)

Recall that, like \( I^i_{i-1} \), \( I^p_i \) is a collection of translated \( B_i \)'s and \( C_i \)'s. So as before, the distance between any two vertices in distinct blocks of \( I^p_i \) is at least \( \alpha_{i+1} - \alpha_i + 1 \). All such \( B_i \)'s and \( C_i \)'s (except for the final copy of \( B_i \)) are contained in translated copies of \( D_i \), which are spaced at intervals of \( h_i \). If we consider some \( x \in I^p_i \), and we add or subtract \( h_i \) to it, we are essentially moving it to the same position within the next or the previous copy of \( D_i \) (so long as this movement does not push, or “scroll”, \( x \) across the \( n_p - 1 \) to 0 gap).

Now, let \( x, y \in I^p_i \) and let \( d \) denote the distance between them along the shorter arc of the circle. Without loss of generality, assume that \( 0 \leq y < x \). Note that \( d = x - y \) or \( d = n_p - (x - y) \), depending on whether or not the shorter arc covers the \( n_p - 1 \) to 0
gap. We must show that $d \not\in S^p_i$, and since $S^p_i \mod h_i = S_i$, it is enough to show that $d' := d \mod h_i$ is not in $S_i$.

We first consider the case where $d = x - y$ (so the shorter arc between $y$ and $x$ does not cover the $n_p - 1$ to 0 gap). We may subtract multiples of $h_i$ from $x$ to give $x' \in I^p_i$ with the property that $0 \leq y \leq x' < y + h_i$. The distance between $x'$ and $y$ is $d' = d \mod h_i$. Now either $x'$ and $y$ are in the same copy of $D_i$ or in consecutive copies of $D_i$. In either case, they are either in the same $B_i$ or $C_i$ block (so, as before, $d' < \alpha_{i-1}$) or in different $B_i$ or $C_i$ blocks (so $d' > \alpha_{i+1} - \alpha_i$). Therefore $d' \not\in S_i$, and we conclude that $x$ and $y$ are not adjacent in $H^p_i$.

We now turn our attention to the more difficult case of $d = n_p - (x - y)$, where the shorter arc between $x$ and $y$ covers the $n_p - 1$ to 0 gap. Here, we may subtract a multiple of $h_i$ from $x$ and $y$ to yield vertices $x'$ and $y'$ where $y'$ is in the first copy of $D_i$ in $I^p_i$ and $x'$ is either in the last copy of $D_i$ (i.e., $x' \in D_i + r_{i,p}h_i$) or in the terminal $B_i$ (i.e., $x' \in B_i + (r_{i,p} + 1)h_i$). Setting $d'' = n_p - (x' - y')$, we consider three possible cases.

Case (a): $d'' < h_i$.

In this case, $d'' = d' := d \mod h_i$. Now, $x'$ and $y'$ are at least as far apart as the last and first elements in $I^p_i$, which, as in the similar case from (ii.b), are at a distance of $\alpha_{i+1} - \alpha_i + 1$. This is too large to be in $S_i$, so $d' \not\in S_i$ and $d \not\in S^p_i$.

Case (b): $x' \in D_i + r_{i,p}h_i$, $d'' \geq h_i$.

In this case we add $h_i$ to $x'$ to get $x''$, but in moving $x'$ across zero, it crosses an extra $B_i$. This extra distance is subtracted from $x'$’s relative position within its $D_i$. This distance
is exactly $n_i$, which is the space between $B_i$'s within $D_i$. Therefore $x''$'s position in the first $D_i$ is exactly one $B_i$ block back\footnote{We do not allow $x'$ to be in the first $B_i$ block of the last $D_i$: If it were, a shift forward by $h_i$ would move it into the terminal $B_i$ in $I^p_i$. This situation is covered separately in Case (c) below.} from the position of $x'$ in the last $D_i$. We still have $x''$ “behind” $y'$ since $d''$ was at least $h_i$, but now both are contained in the first $D_i$ at a distance of $y' - x'' = d' = d \mod h_i$. Now $d'$ is a distance between two elements of $D_i$, and we have seen before that $d' \notin S_i$, so that $d \notin S^p_i$.

Case (c): $x' \in B_i + (r_{i,p} + 1)h_i$, $d'' \geq h_i$.

Write $x' = z + (r_{i,p} + 1)h_i$ where $z \in B_i$, i.e., $z$ is $x'$'s relative position in the terminal $B_i$. Adding $h_i$ to this mod $n_p$ gives

$$z + (r_{i,p} + 1)h_i + h_i - n_p = z + h_i - n_i + [(r_{i,p} + 1)h_i + n_i - n_p]$$
$$= z + h_i - n_i$$
$$= (z - 1) + (k - 2)n_i.$$

So after being moved from $x'$ to $x'' = (x' + h_i) \mod n_p$, we have $x'' \in [(k - 2)n_i + \{-1, 0, \ldots, \alpha_i - 2\}]$. Notice that this set contains the $C_i$ in the first (untranslated) $D_i$, and that $y$ must be in this $C_i$ (otherwise $x'$ and $y'$ would have been closer than $h_i$, putting us in Case (a)). So $x''$ and $y'$ are both in this set $[(k - 2)n_i + \{-1, 0, \ldots, \alpha_i - 2\}]$, and the distance between any two elements in it is at most $\alpha_i - 1 < s$ for all $s \in S_i$, so as before, $d \notin S^p_i$.

Thus in all cases, $d \notin S^p_i$, i.e., no connection distance between points in $I^p_i$ is found in $S^p_i$, so $I^p_i$ must be an independent set of $\langle S^p_i \rangle = H^p_i$.

(iii) $|I^p_i| = \alpha_p$
\[ |I_p^i| = (r_{i,p} + 1)|D_i| + |B_i| \]
\[ = (r_{i,p} + 1)[(k - 2)|B_i| + |C_i|] + \alpha_i \]
\[ = (r_{i,p} + 1)[(k - 2)\alpha_i + \alpha_i - 1] + \alpha_i \]
\[ = (r_{i,p} + 1)[(k\alpha_i - 1) - \alpha_i] + \alpha_i \]
\[ = (r_{i,p} + 1)\alpha_{i+1} - r_{i,p}\alpha_i \]
\[ = (r_{i,p} + 1)(k^i - k^{i-1} - \cdots - 1) - r_{i,p}(k^{i-1} - k^{i-2} - \cdots - 1) \]
\[ = (r_{i,p} + 1)k^i - (2r_{i,p} + 1)k^{i-1} - k^{i-2} - \cdots - 1 \]
\[ = [(r_{i,p} + 1)k - (2r_{i,p} + 1)]k^{i-1} - k^{i-2} - \cdots - 1 \]
\[ = [(k^{p-i} + \cdots + k) - (2k^{p-i-1} + \cdots + 2k + 1)]k^{i-1} - k^{i-2} - \cdots - 1 \]
\[ = [k^{p-i} - k^{p-i-1} - \cdots - k - 1]k^{i-1} - k^{i-2} - \cdots - 1 \]
\[ = k^{p-1} - k^{p-2} - \cdots - k^{i-1} - k^{i-2} - \cdots - 1 \]
\[ = \alpha_p \]

so \(I_p^i\) is of the desired size, even for \(i = p - 1\) \((r = 0)\). Thus we have shown that \(H_i^p\) has an independent set of size \(\alpha_p\), which is exactly what we needed in order to verify that \(\omega_f(H_i^p) < k\), and so we are done. \(\square\)

### 3.3 A Relaxation of Fractional Ramsey Numbers

When we take \(n \rightarrow (k, l)\) to mean that \(K_n = H_1 \oplus H_2\) implies \(\omega(H_1) \geq k\) or \(\omega(H_2) \geq l\), we could equally well write this as “If \(G = H_1 \oplus H_2\) and \(\omega(G) \geq n\), then \(\omega(H_1) \geq k\) or \(\omega(H_2) \geq l\.” We can now fractionalize the entirety of this statement, and take \(z \xrightarrow{\mathbb{C}} (x, y)\)
to mean “If $G = H_1 \oplus H_2$ and $\omega_f(G) \geq z$, then $\omega_f(H_1) \geq x$ or $\omega_f(H_2) \geq y$.” We then define $r^*(x, y)$ to be the infimum of all $z$ for which this statement holds. We may also define the multicolor version of this, where $z \rightarrow^* (x_1, \ldots, x_p)$ means that, if $G = H_1 \oplus \cdots \oplus H_p$ and $\omega_f(G) \geq z$, then $\omega_f(H_i) \geq x_i$ for some $i$. Likewise, $r^*(x_1, \ldots, x_p)$ is the infimum of all $z$ for which $z \rightarrow^* (x_1, \ldots, x_p)$ is true.

To achieve our result, we need the following lemma.

**Lemma 3.6** If $G = H_1 \oplus H_2$, then $\omega_f(G) \leq \omega_f(H_1)\omega_f(H_2)$.

**Proof.** We prove this statement in the form $\chi_f(G) \leq \chi_f(H_1)\chi_f(H_2)$. Let $a_1, a_2, b_1, b_2$ be positive integers such that $\chi_f(H_i) = a_i/b_i = \chi_{b_i}(H_i)/b_i$ for $i = 1, 2$ (the proof of Lemma 1.1 guarantees that such integers exist). Let $c_i$ be a proper $b_i$-fold coloring of $H_i$ using a set of $a_i$ colors, where $c_i(v)$ is the set of $b_i$ colors assigned to $v \in V(G)$, and if $uv \in E(H_i)$ then $c_i(u) \cap c_i(v) = \emptyset$.

We may now construct a $b_1b_2$-fold coloring of $G$ using a set of $a_1a_2$ colors: assign to the vertex $v$ the set of colors $c_1(v) \times c_2(v)$. Now, if $uv \in E(G)$, then $uv \in E(H_1)$ or $uv \in E(H_2)$, which in turn implies that either $c_1(u) \cap c_1(v) = \emptyset$ or $c_2(u) \cap c_2(v) = \emptyset$. This guarantees that $c_1(u) \times c_2(u)$ and $c_1(v) \times c_2(v)$ are disjoint, and so we have a proper $b_1b_2$-fold coloring of $G$. This shows that $\chi_{b_1b_2}(G) \leq a_1a_2$, and so

$$\chi_f(G) = \inf_b \frac{\chi_b(G)}{b} \leq \frac{\chi_{b_1b_2}(G)}{b_1b_2} \leq \frac{a_1a_2}{b_1b_2} = \chi_f(H_1)\chi_f(H_2).$$

**Theorem 3.7** For real numbers $x_1, \ldots, x_p > 2$, we have $r^*(x_1, \ldots, x_p) = x_1x_2 \cdots x_p$. 
Proof. For any $G$ with $\omega_f(G) \geq x_1 \cdots x_p$ and any decomposition $G = H_1 \oplus \cdots \oplus H_p$, if we suppose that $\omega_f(H_i) < x_i$ for all $i$, then Lemma 3.6 (applied repeatedly) implies that $\omega_f(G) < x_1 \cdots x_p$. This is a contradiction, and so we must have $\omega_f(H_i) \geq x_i$ for some $i$, as desired. Therefore $x_1 \cdots x_p \rightarrow (x_1, \ldots, x_p)$, and $r^*(x_1, \ldots, x_p) \leq x_1 \cdots x_p$.

We prove the lower bound by inducting on $p$.

**BASE** $\bullet$ $p = 1$:

That $r^*(x) = x$ is immediate.

**INDUCTION HYPOTHESIS** $\bullet$ Suppose that $r^*(x_1, \ldots, x_{p-1}) = x_1 \cdots x_{p-1}$:

Take any $z < x_1 \cdots x_p$. We must find a graph $G$ and a decomposition $G = H_1 \oplus \cdots \oplus H_p$ where $\omega_f(G) \geq z$ and $\omega_f(H_i) < x_i$ for all $i$. To start, we choose rational $q_1$ and $q_2$ such that

$$2 < q_1 < x_1 \cdots x_{p-1}, \quad 2 < q_2 < x_p \quad \text{and} \quad q_1 q_2 \geq z.$$

By our induction hypothesis, there is a graph with decomposition $G_{p-1} = H'_1 \oplus \cdots \oplus H'_{p-1}$ such that $\omega_f(G_{p-1}) \geq q_1$ but $\omega_f(H'_i) < x_i$ for each $i = 1, \ldots, p - 1$. Recalling that $\omega_f(C_{(q_2)}) = q_2$, we take $G = C_{(q_2)}[G_{p-1}]$ (wherein each vertex of $C_{(q_2)}$ is replaced with a copy of $G_{p-1}$), so we have $\omega_f(G) \geq q_1 q_2 \geq z$ by Lemma 1.5. Let $C_{(q_2)}$ have $m$ vertices, and $G_{p-1}$ have $n$ vertices. We may partition the edges of $G$ into the set of edges *within* copies of $G_{p-1}$, and the set of edges *between* copies of $G_{p-1}$. The former yields a graph of the form

$$
\overline{K_m}[G_{p-1}] = \overline{K_m}[H'_1 \oplus \cdots \oplus H'_{p-1}] = \overline{K_m}[H'_1] \oplus \cdots \oplus \overline{K_m}[H'_{p-1}],
$$

while the later is $C_{(q_2)}[\overline{K_n}]$. So if we set $H_i = \overline{K_m}[H'_i]$ for $i = 1, \ldots, p - 1$, and $H_p =$
$C_{(q_2)}[K_n]$, then $G = H_1 \oplus \cdots \oplus H_p$. Further,

$$\omega_f(H_i) = \omega_f(K_n)\omega_f(H'_i) = \omega_f(H'_i) < x_i \quad \text{for } i = 1, \ldots, p - 1$$

$$\omega_f(H_p) = \omega_f(C_{(q_2)})\omega_f(K_n) = \omega_f(C_{(q_2)}) = q_2 < x_p$$

This construction shows that $z \rightarrow^* (x_1, \ldots, x_p)$ is false, and so $r^*(x_1, \ldots, x_p) \geq x_1x_2 \cdots x_p$.

Since this relation is implied by our induction hypothesis, we are done. \(\square\)

### 3.4 $b$-Ramsey Numbers

We may define the $b$-Ramsey number of a graph by replacing $\omega$ in the Ramsey definition with $\omega'_b$ instead of $\omega_f$. Recall from Section 1.1 that

$$\omega'_b(G) = \max 1 \cdot y \text{ s.t. } M' \cdot y \leq 1, \quad y \in \{0, \frac{1}{b}, \frac{2}{b}, \ldots, \frac{b}{b}\}^n.$$ 

A $b$-clique of $G$ is a function $g_b : V(G) \rightarrow \{0, \frac{1}{b}, \frac{2}{b}, \ldots, \frac{b}{b}\}$ such that $\sum_{v \in I} g_b(v) \leq 1$ for all $I \in \mathcal{I}$. The weight, or value, of this $b$-clique is $w(g_b) = \sum_{v \in V(G)} g_b(v)$, and $\omega'_b(G)$ is the minimum\(^{15}\) value of $w(g_b)$ taken over all $b$-cliques of $G$.

We let $n \rightarrow^b (x, y)$ stand for the statement “If $K_n = H_1 \oplus H_2$, then $\omega'_b(H_1) \geq x$ or $\omega'_b(H_2) \geq y$.” Then the $b$-Ramsey number $r_b(x, y)$ is the least positive integer $n$ for which this statement is true.

Because $\omega_f(G) \geq \omega'_b(G) \geq \omega(G)$ for any positive integer $b$, it immediately follows that $r_f(x, y) \leq r_b(x, y) \leq r(x, y)$. In general, because $r_b(x, y)$ involves a discrete optimization invariant similar to $r(k, l)$, we expect computation of these values to be difficult (as opposed to the relative ease of calculating $r_f(x, y)$). We do, however, have two principle results

---

\(^{15}\)Since we are not discussing infinite graphs, we do mean “minimum” and not “infimum”.

regarding the limiting behavior of $r_b$. The first of these tells us that $r_2(k, k)$, like $r(k, k)$, grows exponentially in $k$, although the bound achieved on this growth rate is considerably smaller.

**Lemma 3.8** The edge set of $K_n$ contains at least $\frac{1}{6}(n - 3)(n - 4)$ edge-disjoint triangles.

**Proof.** It is a long-known result\(^{16}\) that there is a Steiner triple system on $n$ elements iff $n \equiv 1$ or $3 \mod 6$. In other words, for sets of size $n \equiv 1$ or $3 \mod 6$, we may find a collection $T$ of size 3 subsets such that every pair of elements appears in exactly one member of $T$. This is equivalent to saying that $T$ partitions $E(K_n)$ into triangles. Such a partition necessarily has $n(n - 1)/6$ triangles. Even if $n \equiv 0 \mod 6$, we may still partition the edges of a $K_{n-3}$ subgraph into $(n - 3)(n - 4)/6$ triangles, giving the desired result. $\square$

Note that, while the above value is not always best possible, it still approaches $n^2/6$ asymptotically, which clearly is the best possible limit.

**Theorem 3.9** If positive integers $n$ and $k$ satisfy

$$2 \left( \frac{n}{k} \right) \left( \frac{7}{8} \right)^{(k-3)(k-4)/6} < 1,$$

then $r_2(k, k) > n$.

**Proof.** We start by defining

$$\bar{\omega}_2(G) = \max \{ 1 \cdot y \mid M'y \leq 1, \ y \in \{0, \frac{1}{2}\}^n \},$$

which is identical to $\omega'_2(G)$, except that we are only allowed to assign weights 0 or 1/2, and not 1, to vertices. We refer to feasible solutions to this program as *half-cliques*. Note

\(^{16}\)First shown by Kirkman around 1850; see [1].
that, in any such half-clique, the set of vertices receiving weight 1/2 must be triangle-free, and in fact, \( \bar{\omega}_2(G) \) is 1/2 times the number of vertices in the largest triangle-free induced subgraph of \( G \). Now, the set of positively weighted vertices in any 2-clique must also be triangle-free, but here we are using weights 1/2 and 1. Multiplying \( \omega_2(G) \) by 2 is equivalent to assigning weight 1 to every vertex in the largest triangle-free subgraph, so we must have \( \omega'_2(G) \leq 2\bar{\omega}_2(G) \).

We now employ a probabilistic technique similar to those used to calculate lower bounds for ordinary Ramsey numbers (see, for instance, [13]). Fix \( n \), and let us red/blue color the edges of \( K_n \), giving any edge red or blue with probability 1/2, independent of the coloring of any other edges. For any size \( k \) subset \( S \) of \( V(K_n) \), define the events

\[
A_S = \{ E(S) \text{ has no blue triangle} \},
\]

\[
B_S = \{ E(S) \text{ has no red triangle} \},
\]

\[
B = \{ K_n \text{ contains a monochromatic weight } k/2 \text{ half-clique} \}
\]

\[
= \bigcup_{|S|=k} (A_S \cup B_S).
\]

Now, the probability that any given triangle is not all blue is 7/8, and any size \( k \) set \( S \) contains at least \((k-3)(k-4)/6\) edge-disjoint triangles by Lemma 3.8, so \( \Pr\{A_S\} = \Pr\{B_S\} < (7/8)^{(k-3)(k-4)/6} \). So we have

\[
\Pr\{B\} = \Pr\left\{ \bigcup_{|S|=k} (A_S \cup B_S) \right\} \leq 2 \sum_{|S|=k} \Pr\{A_S\} < 2 \left( \begin{array}{c} n \\ k \end{array} \right) \left( \frac{7}{8} \right)^{(k-3)(k-4)/6}.
\]

If we choose \( n \) small enough so as to make this quantity less than 1, then \( \Pr\{B^c\} > 0 \), so there must exist some edge 2-coloring of \( K_n \) with no monochromatic half-cliques of
weight $k/2$. That is, there is some $K_n = H_1 \oplus H_2$ with $\omega_2(H_i) < k/2$ for $i = 1, 2$. Since $\omega'_2(H_i) \leq 2\omega_2(H_i)$, we know that $n \rightarrow (k, k)$ is false. □

Let us perform a rough asymptotic analysis of $n$ versus $k$ to find a lower bound on $r_2(k, k)$. If we take \( \binom{n}{k} \approx n^k \) and \( (7/8)^{(k-3)(k-4)/6} \approx (7/8)^{k^2/6} \), then

\[
2 \left( \frac{n}{k} \right) \left( \frac{7}{8} \right)^{\lfloor k(k-1)/6 \rfloor} < 1 \quad \text{becomes} \quad 2n^k \left( \frac{7}{8} \right)^{k^2/6} < 1.
\]

Then $2n^k < (8/7)^{k^2/6}$, which gives, approximately, $n < (8/7)^{k/6}$. So we have the approximate bound of $r_2(k, k) > \left((8/7)^{1/6}\right)^k \approx 1.0225^k$. While this is not nearly as good as the $\sqrt{2}^k$ lower bound for $r(k, k)$, we do know that $r(k, k) > r_2(k, k)$ in general, and this does at least establish the exponential growth of $r_2(k, k)$.

Since $\omega'_b(G) \rightarrow \omega_f(G)$ as $b \rightarrow \infty$ for any graph $G$ (Theorem A.2), we wonder if the same holds for $b$- and fractional Ramsey numbers, i.e. does $r_b(x, y) \rightarrow r_f(x, y)$? Note that, since $r_b$ and $r_f$ are integer valued, convergence occurs iff there exists a $B \in \mathbb{N}$ for which $b \geq B$ implies $r_b = r_f$.

**Theorem 3.10** $r_b(x, y) \rightarrow r_f(x, y)$ as $b \rightarrow \infty$ if and only if $(x, y)$ is not a discontinuity point of the function $r_f$.

**Proof.** Recall $x, y, k, l, \varepsilon, \delta$ and $q = \min\{\lceil \varepsilon l \rceil, \lceil \delta k \rceil\}$ of Theorem 3.1. Notice that if we hold $k$ and $l$ fixed, $\varepsilon$ and $\delta$ may range freely between multiples of $1/l$ and $1/k$, respectively, without changing the value of $q$. Thus $r_f(x, y)$ is constant over these rectangular regions of the plane. These rectangles are closed along their upper and right edges, and open on their lower and left edges. All discontinuity points of $r_f(x, y)$ lie on these edges, though not all
points on these edges are discontinuity points. To be more precise, \((x, y)\) is a discontinuity point of \(r_f\) if \(r_f(x + \epsilon, y + \epsilon) > r_f(x, y)\) for all \(\epsilon > 0\).

\(\Rightarrow\) Lemma 3.11 If \((x, y)\) is a discontinuity point of \(r_f\) with \(r_f(x, y) = n\), then there exists a decomposition \(K_n = H_1 \oplus H_2\) such that \(\omega_f(H_1) \leq x\) and \(\omega_f(H_2) \leq y\).

**Proof.** Suppose to the contrary. Then for every such decomposition, either \(\omega_f(H_1) > x\) or \(\omega_f(H_2) > y\). There are only a finite number of such decompositions, so let \(\epsilon = \min\{\omega_f(H_1) - x, \omega_f(H_2) - y\}\), taken over all such positive values for all such decompositions. Thus for any decomposition \(K_n = H_1 \oplus H_2\), we know that \(\omega_f(H_1) \geq x + \epsilon\) or \(\omega_f(H_2) \geq y + \epsilon\). But this tells us that \(r_f(x + \epsilon, y + \epsilon) = n\), contradicting the fact that \((x, y)\) is a discontinuity point of \(r_f\). Our supposition to the contrary is therefore false. \(\square\)

For discontinuity point \((x, y)\), take the decomposition \(K_n = H_1 \oplus H_2\) indicated by Lemma 3.11. For \(i = 1, 2\), let \(\omega_f(H_i) = c_i/d_i\), where \(c_i/d_i\) is a lowest terms fraction. Let \(b\) be any positive integer such that neither \(d_1\) nor \(d_2\) divide \(b\). Then there is no integer \(a_i\) such that \(a_i/b = c_i/d_i\). Since \(\omega_b'(G)\) must always be a multiple of \(1/b\), we cannot have \(\omega_b'(H_i) = c_i/d_i\). Thus

\[
\omega_b'(H_1) < \omega_f(H_1) \leq x \quad \text{and} \quad \omega_b'(H_2) < \omega_f(H_2) \leq y
\]

and so \(r_b(x, y) > n = r_f(x, y)\). Since we may choose such \(b\) to be arbitrarily large, we have \(r_b \not\rightarrow r_f\) at this \((x, y)\).

\(\Leftarrow\) Lemma 3.12 For any graph \(G\) and \(b \in \mathbb{N}\), \(\omega_b'(G) > \omega_f'(G) - |V(G)|/b\).
Proof. Start with a maximum fractional clique, and round all weights on vertices down to the nearest multiple of $1/b$. The result is a valid $b$-clique, and total weight less than $|V(G)|/b$ has been removed. \[\square\]

Let $r_f(x, y) = n$ be constant over all $x \in (x_1, x_2], y \in (y_1, y_2]$. Choose any specific $x \in (x_1, x_2)$ and $y \in (y_1, y_2)$, so that $(x, y)$ is not a discontinuity point of $r_f$. Let $d = \min\{x_2 - x, y_2 - y\} > 0$, and choose $B \in \mathbb{N}$ such that $n/B < d$. Now, for any decomposition $K_n = H_1 \oplus H_2$, either $\omega_f(H_1) \geq x_2$ or $\omega_f(H_2) \geq y_2$. Then for any $b \geq B$, either

$$\omega_b(H_1) > \omega_f(H_1) - n/b \geq x_2 - d \geq x_2 - (x_2 - x) = x$$

or

$$\omega_b(H_2) > \omega_f(H_2) - n/b \geq y_2 - d \geq y_2 - (y_2 - y) = y,$$

so $n \xrightarrow{b} (x, y)$. Thus $r_b(x, y) = r_f(x, y)$ for all $b \geq B$, and so $r_b \rightarrow r_f$ at $(x, y)$. \[\square\]

Clearly, there is still much work that could be done in the area of $b$-Ramsey numbers. Even though $r_b(x, x)$ (presumably) grows exponentially in $x$, for almost any fixed $x$ we have $r_b(x, x) \rightarrow r_f(x, x)$, a function which only grows quadratically in $x$. There seems to be some interesting ground to cover in comparing the growth rates of $r_b(x, y)$ in $x$ and $y$ versus $b$.

### 3.5 Lovász-$\theta$ Ramsey Numbers

The fractional clique number of a graph is a relaxation of the ordinary clique number; the Lovász-$\theta$ number of a graph, denoted $\theta(G)$, represents a weaker relaxation of clique...
number. That is,
\[ \omega(G) \leq \vartheta(G) \leq \omega_f(G) \leq \chi(G) \]
for any finite graph \( G \). \( \vartheta(G) \) was introduced by Lovász in 1977 (see [10]), and has been well studied since then (see [7] for an overview of history and results). We will define Lovász-\( \vartheta \) Ramsey numbers by replacing clique number with the Lovász-\( \vartheta \) number\(^{17}\).

To define \( \vartheta(G) \), we first need to define orthogonal labelings. An orthogonal labeling of a graph \( G = (V, E) \) is an assignment of a unit\(^{18}\) vector \( a_v \) to each \( v \in V \) such that \( a_u \cdot a_w = 0 \) whenever \( uv \in E \). That is, adjacent vertices are assigned perpendicular vectors, and \( ||a_v|| = 1 \) for all \( v \in V \). These vectors may be of any fixed dimension \( d \). The cost of a vector in such a labeling is defined to be \( c(a_v) = a_{1v}^2 \), where \( a_{1v} \) is the first entry of \( a_v \). The cost of the labeling, denoted \( c(a) \), is just the vector of costs (whose \( v \)-th entry is \( c(a_v) \)).

Recall the integer (ID) and linear (DP) duals defining \( \omega \) and \( \omega_f \) from Section 1.1. Let us refer to the feasible regions of these programs as \( \Omega(G) \) and \( \Omega_f(G) \), respectively. We now define the region
\[ \Theta(G) = \{ y \in \mathbb{R}^n : c(a) \cdot y \leq 1 \text{ for all orthogonal labelings } a \text{ of } G, \ y \geq 0 \}. \]
Similarly to \( \omega \) and \( \omega_f \), we define
\[ \vartheta(G) = \max 1 \cdot y \text{ s.t. } y \in \Theta(G). \]
It is easy to show\(^{19}\) that \( \Omega(G) \subseteq \Theta(G) \subseteq \Omega_f(G) \), from which \( \omega(G) \leq \vartheta(G) \leq \omega_f(G) \)

\(^{17}\)Note: Our definitions of \( \vartheta \) and orthogonal labelings of \( G \) are more commonly taken to be those of \( \overline{G} \). We switch the roles of these two quantities to maintain consistency with our previous work. As Ramsey numbers are symmetric in the usage of \( G \) and \( \overline{G} \), our choice of definitions will not affect our Ramsey results.

\(^{18}\)Requiring unit vectors is not standard in the definition of orthogonal labelings, but is done without loss of generality for our purposes; see [7].

\(^{19}\)See Lemma 2 from [7].
immediately follows.

We now take \( n \rightarrow^{\vartheta} (x, y) \) to mean that, whenever \( K_n = H_1 \oplus H_2 \), we must have \( \vartheta(H_1) \geq x \) or \( \vartheta(H_2) \geq y \). Then the \( \vartheta \)-Ramsey number \( r_\vartheta(x, y) \) is the least positive integer \( n \) for which \( n \rightarrow^{\vartheta} (x, y) \). While we cannot compute \( r_\vartheta \) exactly as we did with \( r_f \), we can show that it’s value is nearly the same as \( r_f \). Our result follows easily from the following lemma\(^{20}\).

**Lemma 3.13** For a graph \( G \) on \( n \) vertices, \( \vartheta(G)\overline{\vartheta(G)} \geq n \); if \( G \) is vertex transitive, then equality holds. \( \square \)

**Theorem 3.14** \( r_f(x, y) \leq r_\vartheta(x, y) \leq \lceil xy \rceil \).

**Proof.** Fix \( x, y \geq 2 \). Let \( n = r_\vartheta(x, y) \), and let \( K_n = H_1 \oplus H_2 \) be any edge 2-coloring of \( K_n \). Then \( \omega_f(H_1) \geq \vartheta(H_1) \geq x \) or \( \omega_f(H_2) \geq \vartheta(H_2) \geq y \), and so \( n \rightarrow^{f} (x, y) \). Thus it follows that \( r_f(x, y) \leq r_\vartheta(x, y) \).

Now take \( n = \lceil xy \rceil \), and any decomposition \( K_n = H_1 \oplus H_2 \) (with \( H_1 = \overline{H_2} \)). Either \( \vartheta(H_1) \geq x \) or \( \vartheta(H_1) < x \) implies that \( \vartheta(H_2) \geq \frac{n}{\vartheta(H_2)} > \frac{xy}{x} = y \) by Lemma 3.13. Thus \( r_\vartheta(x, y) \leq \lceil xy \rceil \). \( \square \)

Since \( r_f(x, y) \) is very nearly as large as \( xy \), we have fairly tight bounds on \( r_\vartheta(x, y) \). It is interesting to note the closeness of the values of \( r_\vartheta \) to \( r_f \), even though \( \vartheta \) lies somewhere between \( \omega \) and \( \omega_f \).

Note that we did not use the second part of Lemma 3.13. If we could establish values of \( \vartheta \) for a large class of vertex transitive graphs (as we did with \( C_{n,m} \) for \( \omega_f \)), we could likely achieve even more accurate bounds on \( r_\vartheta \).

\(^{20}\)See Lemma 23 and Theorem 25 from [7].