1 Gödel’s Completeness Theorem

We will begin our study of logic by showing Gödel’s Completeness theorem, which is a fundamental theorem in mathematical logic establishing a correspondence between semantic truth and syntactic provability. This relationship is foundational in model theory, as it allows us to use the existence of models as justification for mathematical statements.

1.1 Propositions and Connectives

Formal logic is the study of reasoning. In order to study reasoning with mathematical rigor, we must carefully choose the language in which we express our ideas.

We will begin by studying symbols and focusing on constructing strings of symbols in a certain way. Later on we will examine what these symbols mean. That is, we will begin by studying syntax and later we will give our strings of symbols semantic meaning.

Definition 1. The language of propositional logic consists of the following alphabet

1. proposition symbols/variables: \( p_0, p_1, p_2, \ldots \)
2. logical constant: \( \bot \)
3. connectives: \( \land, \lor, \rightarrow, \leftrightarrow, \neg \)
4. auxiliary symbols: (, )

The connectives have the following common names:

| \( \land \) | and | conjunction |
| \( \lor \) | or | disjunction |
| \( \rightarrow \) | if \ldots, then | implication |
| \( \neg \) | not | negation |
| \( \leftrightarrow \) | iff | equivalence, bi-implication |
| \( \bot \) | falsity | absurdum |

Definition 2. The set of propositions, denoted \( \text{PROP} \), is the smallest set \( X \) with the following properties:

(i) \( p_i \in X \) for \( i \in \mathbb{N}, \bot \in X \)
(ii) $\phi, \psi \in X \Rightarrow (\phi \Box \psi) \in X$ where $\Box \in \{\land, \lor, \leftarrow, \leftrightarrow\}$

(iii) $\phi \in X \Rightarrow (\neg \phi) \in X$

Observations:

- We use the single line arrows ($\rightarrow, \leftrightarrow$) for connectives and double lined arrows ($\Rightarrow, \Leftrightarrow$) when working in the *meta-language*, that is, when making statements about propositions as in (ii) and (iii), not within propositions themselves.

- We will see that it is important that we specify that PROP is the *smallest* set of strings of symbols with these properties. This will be useful for proving if a given string of symbols is in PROP. Another way we can view this is as the intersection of all sets satisfying these three properties. In other words, let $B$ be the collection of all sets $X$ satisfying conditions (i), (ii), and (iii). Then $PROP = \bigcap_{X \in B} X$. Observe that PROP will satisfy conditions (i), (ii), and (iii), since for all $X \in B$, $p_i \in X$ and $\bot \in X$, and given $\phi, \psi \in PROP$, $\phi, \psi$ are in every set satisfying (i)-(iii), so by (ii) $(\phi \Box \psi)$ is in every such set, and by (iii), $(\neg \phi)$ is every such set, and thus, their intersection. Finally, note that $B \neq \emptyset$, since the the collection of all finite strings of elements of the alphabet satisfies conditions (i)-(iii).

- $\bot$ is not quite a “connective” in that it cannot be used to connect other propositions. We may also refer to it as a “propositional constant”.

- Similarly, note that $\neg$ is a *unary* connective, in that it is applied to a single proposition, while $\land, \lor, \leftarrow, \leftrightarrow$ are *binary* connectives. This is why we handle $\neg$ in a separate case.

- The propositions in item (i) are called *atoms* or *atomic propositions*.

- We use greek symbols like $\phi$ and $\psi$ as *variables for propositions*. These are in the *meta-language*, in that they are not themselves propositions.

Examples:

$$(p_8 \rightarrow p_6) \in PROP$$

$$((\bot \lor p_{75}) \land (\neg p_3)) \in PROP$$

We can check these are in PROP by carrying out the construction of these sentences according to the rules in the definition. We will see more of this soon.

Non-Examples:

$$((\rightarrow \land \not\in) \notin PROP$$

$$p_0 \leftrightarrow p_9 \notin PROP$$

$$\neg\neg \bot \notin PROP$$
It seems reasonably obvious that the first example is not in PROP. For the latter two it is a little more subtle, especially if we are relying on semantic intuition (these appear to be reasonable mathematical statements). If we carefully examine the rules of PROP, introducing connectives like → and ¬ requires the introduction of parentheses (later on we will agree on shorthand representations of propositions which do not require so many brackets). This however (for now) is not sufficient to prove that these are not in PROP.

We only have one tool at our disposal so far to prove facts like this: that PROP is the smallest set of strings of symbols with properties (i)-(iii). By smallest we mean with respect to the subset relation. Another way to say this is that there is no strict subset of PROP which satisfies conditions (i)-(iii) and any set satisfying conditions (i)-(iii) will contain PROP.

We will prove ¬¬⊥ \notin PROP.

**Proof.** Suppose for the sake of contradiction that it is.

Let \( Y = PROP \setminus \{\neg\neg\bot\} \). To arrive at a contradiction, we will show that \( Y \) satisfies (i), (ii), and (iii), contradicting that PROP is the smallest set to do so. Since \( p_i \) for \( i \in \mathbb{N} \) and \( \bot \) are in PROP and are not equal to \( \neg\neg\bot \), they are in \( Y \), so (i) is satisfied.

For (ii), let \( \phi, \psi \in Y \) be given. Then since they are in PROP, \( (\phi \Box \psi) \in PROP \) for \( \Box \in \{\land, \lor, \leftarrow, \leftrightarrow\} \). Since \( \neg\neg\bot \) is not of the form \( (\phi \Box \psi) \) for any possible \( \phi, \psi \), and \( \Box \) they cannot be equal, so \( (\phi \Box \psi) \in Y \).

Similarly for (iii), let \( \phi \in Y \) be given. Since \( \phi \in PROP \), \( (\neg\phi) \in PROP \). Since \( \neg\neg\bot \) is not of the form \( (\neg\phi) \) for any possible choice of \( \phi \), \( (\neg\phi) \in Y \), as required.

So \( Y \) satisfies (i), (ii), and (iii). \( \Rightarrow\Leftarrow \). 

Note that equality is a relation in the meta-language, and we consider two strings to be equal if they are literally the same exact string. Remember, our focus right now is on syntax, not semantics. Later on we will study the situation in which two strings of symbols are logically equivalent, or mean the same thing (semantics).

### 1.1.1 Induction on Propositions

One of the main advantages of our restrictions on propositions is that we can use induction to prove things about the set of all propositions in the same way that we use induction to prove facts about the set of all natural numbers.

In mathematical induction, we prove a statement is true about some base case (usually 0) and then prove that if the fact is true about some \( n \), it is necessarily true about \( n + 1 \). In this way we can cover all of the natural numbers.

When we induct on formulas, atomic propositions (propositional variables and \( \bot \)) will be our base case, and for our induction hypothesis we will assume the fact about some propositions \( \phi \) and \( \psi \) and show that it is necessarily true about \( (\phi \Box \psi) \) and \( (\neg\phi) \).

<table>
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<th>PROPOSITION</th>
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<td>Base Case</td>
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<td>( P(n + 1) ) ( A((\phi \Box \psi)), A((\neg\phi)) )</td>
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**Theorem 1 (Induction Principle).** Let \( A \) be a property of strings of symbols from our propositional alphabet. \( A(\phi) \) holds for all \( \phi \in PROP \) if

(i) \( A(p_i) \) for all \( i \in \mathbb{N} \) and \( A(\bot) \),
(ii) \( A(\phi), A(\psi) \Rightarrow A((\phi \Box \psi)) \),

(iii) \( A(\phi) \Rightarrow A(\neg \phi) \).

**Proof.** Assume these three conditions hold and let \( X = \{ \phi \in PROP | A(\phi) \} \), so \( X \subset PROP \). By definition, \( X \) satisfies items (i), (ii), and (iii) in the definition of PROP. So since PROP is the smallest set with these properties, \( X = PROP \).

**Side note:** This proof only relies on PROP being the smallest set of strings of symbols satisfying (i), (ii), and (iii). The principle of mathematical induction on the natural numbers is justified in a similar way, and can be stated precisely as

\[
\forall X \subset \mathbb{N}(0 \in X \land (\forall n \in \mathbb{N}(n \in X \Rightarrow n + 1 \in X)) \Rightarrow X = \mathbb{N}).
\]

In words, this is saying that for any subset of \( \mathbb{N} \) (e.g., the collection of elements satisfying some property), if \( 0 \) is in that set and for every \( n \) in the set, its successor is in the set, then the set is just all of \( \mathbb{N} \) (that is, the property is true for all of \( \mathbb{N} \)). This is often stated as an axiom, for example, it is an axiom (scheme) of Peano Arithmetic. Alternatively, it follows from the assumption that \( \mathbb{N} \) is well-ordered (there is no infinite strictly descending chain of natural numbers). To see this, assume \( P \) is a property of natural numbers such that \( P(0) \) holds and for all \( n \in \mathbb{N} \), \( P(n) \Rightarrow P(n + 1) \). Then let \( S = \{ m \in \mathbb{N} | P(m) \text{ is false} \} \). If \( S \neq \emptyset \), by the well-ordering principle, there exists \( n \in S \) which is minimal. \( n \neq 0 \) since \( P(0) \) is true, so let \( m = n - 1 \). Since \( n \) is minimal, \( P(m) \) must be true, but then \( P(m + 1) = P(n) \) is true. \( \Rightarrow \Leftarrow \).

As is the case with natural numbers, we can use the induction principle as a tool to prove things about PROP, for example,

**Fact 1.** Every proposition has an even number of brackets.

**Proof.** By induction on \( \phi \).

Base Case/(i): If \( \phi \) is \( p_i \) for some \( i \in \mathbb{N} \) or \( \bot \), it has 0 brackets (which is even).

Induction Hypothesis: Let \( \phi, \psi \in PROP \) and assume \( \phi \) has \( 2n \) brackets and \( \psi \) has \( 2m \) brackets for some \( n, m \in \mathbb{N} \).

(ii): \( (\phi \Box \psi) \) has \( 1 + 2n + 2m + 1 + 1 = 2(n + m + 1) \) brackets, which is even.

(iii): \( (\neg \phi) \) has \( 1 + 2n + 1 = 2(n + 1) \) brackets, which is even.

Hence, every proposition has an even number of brackets. \( \square \)

Thinking of PROP as the smallest set satisfying some properties is not the most intuitive way to think about what the set of propositions actually *is*. In the same way, we could define \( \mathbb{N} \) as the smallest set \( X \) such that \( 0 \in X \) and \( n \in X \Rightarrow n + 1 \in X \) (while this is true, it is not especially natural... pun intended). It is much more intuitive to think about constructing \( \mathbb{N} \) by starting at 0 and adding 1 indefinitely. We can do something analogous with PROP.

**Definition 3.** A sequence \( \phi_0, \ldots, \phi_n \) is called a *formation sequence* of \( \phi \) if \( \phi_n = \phi \) and for all \( i \leq n \), either

- \( \phi_i \) is atomic, or

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\( \frac{\text{Page 4}}{\text{Document}} \)
• \( \phi_i = (\phi_j \Box \phi_k) \) for some \( j, k < i \), or
• \( \phi_i = (\neg \phi_j) \) for some \( j < i \).

Examples: Let \( \phi \) be \( ((p_1 \land p_0) \to p_2) \). The following are formation sequences of \( \phi \):

• \( p_0, p_1, p_2, (p_1 \land p_0), ((p_1 \land p_0) \to p_2) \)
• \( p_0, p_1, (p_1 \land p_0), p_2, ((p_1 \land p_0) \to p_2) \)

Note that the order can change, as long as each entry is either atomic or is built from previous entries using connectives.

• \( p_0, p_1, p_2, p_3, p_{100}, (\neg p_{100}), ((\neg p_{100}) \lor p_{100}), (p_1 \land p_0), (p_3 \to p_{100}), p_0, p_0, p_0, ((p_1 \land p_0) \to p_2) \)

There may be redundant or irrelevant entries in a formation sequence.

Observation: The first element of a formation sequence must always be atomic since \( \phi_0 \) must satisfy one of the three conditions in the definition, and the second and third are impossible since there are no indices \( j, k < 0 \).

Lemma 2. Every proposition has a formation sequence.

Proof. By induction on \( \phi \).

(i) If \( \phi \) is atomic, then the sequence consisting just of \( \phi \) is a formation sequence of \( \phi \).

Now let \( \phi \) and \( \psi \) be propositions with formation sequences \( \phi_0, \ldots, \phi_n \) and \( \psi_0, \ldots, \psi_m \) respectively.

(ii) \( \phi_0, \ldots, \phi_n, \psi_0, \ldots, \psi_m, (\phi_n \Box \psi_m) \) is a formation sequence for \( (\phi \Box \psi) \). First note that \( (\phi \Box \psi) \) is equal to \( (\phi_n \Box \psi_m) \) since \( \phi_n = \phi \) and \( \psi_m = \psi \). Since \( \phi_0, \ldots, \phi_n \) and \( \psi_0, \ldots, \psi_m \) are formation sequences, concatenating them will still satisfy the three conditions in the definition, and \( (\phi_n \Box \psi_m) \) satisfies the second condition.

(iii) \( \phi_0, \ldots, \phi_n, (\neg \phi_n) \) is a formation sequence for \( (\neg \phi) \), since \( (\neg \phi) = (\neg \phi_n) \) because \( \phi = \phi_n \), and \( (\neg \phi_n) \) satisfies the third condition in the definition of formation sequence.

In fact, it turns out the PROP is exactly the set of strings of symbols with formation sequences. This is a more intuitive way to think about propositions, and we will prove this equivalence by induction on \( \phi \).

Theorem 3. PROP is the set of all expressions with formation sequences.

Proof. Let \( F \) be the set of all strings of symbols with formations sequences. We have shown that \( PROP \subset F \). We will show that \( F \subset PROP \).

Let \( \phi \in F \) be given and let \( \phi_0, \ldots, \phi_n \) be its formation sequence. We will show by (strong) induction on \( n \) that \( \phi_0, \ldots, \phi_n \) are all in PROP, and thus, since \( \phi = \phi_n \), \( \phi \in PROP \).

For \( n = 0 \), \( \phi_0 \) must be atomic, so \( \phi_0 \) is in PROP.

Let \( k \leq n \) be given and assume for all \( j < k \) that \( \phi_j \in PROP \).
If $\phi_k$ is atomic, $\phi_k \in PROP$.

(ii) If $\phi_k = (\phi_j \square \phi_l)$ for some $j, l < k$, then by IH $\phi_j$ and $\phi_l$ are in PROP, so by (ii) in the definition of PROP, $\phi_k \in PROP$.

(iii) If $\phi_k = (\neg \phi_j)$ for some $j < k$, then by IH $\phi_j \in PROP$, so by (iii) in the definition of PROP, $\phi_k \in PROP$.

1.1.2 Definition by Recursion

Now that we have a characterization of PROP using induction, we can define functions on PROP by recursion.

We have seen many examples of this in other parts of math and computer science, for example, we define factorials by $0! = 1$ and $(n+1)! = (n+1) \cdot n!$. There is more happening under the hood though, and this can be described precisely as follows:

**Theorem 4** (Definition by Recursion for $\mathbb{N}$). Let $A$ be a set and let mappings $H_z(0) \in A$ and a mapping $H_s : \mathbb{N} \times A \to A$ be given. Then there exists exactly one function $F : \mathbb{N} \to A$ satisfying the following:

- $F(0) = H_z(0)$
- $F(n+1) = H_s(n+1, F(n))$

Though we often take this for granted, this is a deep fact about $\mathbb{N}$ and requires careful justification using induction. In the above example of factorial, $A = \mathbb{N} H_z(0) = 1$ and $H_s(n, m) = n \cdot m$. The theorem tells us that there is a unique function $F : \mathbb{N} \to \mathbb{N}$ such that $F(0) = H_z(0) = 1$ and $F(n+1) = H_s(n+1, F(n)) = (n+1) \cdot F(n)$. We call this function factorial, and denote it $n! = F(n)$.

Now that we have an induction principle for PROP, we can do an analogous thing to define functions on PROP.

**Theorem 5** (Definition by Recursion for PROP). Let $A$ be a set, and let the following mappings be given:

- $H_{at} : \{ \phi \in PROP| \phi \text{ is an atom} \} \to A$
- $H_\square : PROP \times A^2 \to A$
- $H_\neg : PROP \times A \to A$

Then there exists exactly one function $F : PROP \to A$ such that

- $F(\phi) = H_{at}(\phi)$ for $\phi$ atomic,
- $F((\phi \square \psi)) = H_\square((\phi \square \psi), F(\phi), F(\psi))$
- $F((\neg \phi)) = H_\neg((\neg \phi), F(\phi))$. 

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Proof. (Outline.) Again, this proof requires induction. We leave it as an exercise to the reader, but we will outline the steps here:

• Let \( H_{at}, H_{\Box}, \) and \( H_{\neg} \) be as in the theorem. Let \( F^* \subset \text{PROP} \times A \) be defined inductively as follows: if \( \phi \) is atomic, \((\phi, H_{at}(\phi)) \in F^*\). Let \( \phi \) and \( \psi \) be such that there are some \( a, b \in A \) such that \((\phi, a) \in F^*\) and \((\psi, b) \in F^*\). Then we require that \(( (\phi \Box \psi), H_{\Box}((\phi \Box \psi), a, b) ) \in F^*\) and \(( (\neg \phi), H_{\neg}((\neg \phi), a) \in F^*\).

• Since PROP is the set of expressions with formation sequences, we see that for all \( \phi \) in PROP there exists \( a \in A \) such that \( (\phi, a) \in F^*\).

• Show that \( F^* \) is a function by showing that if \( (\phi, a), (\phi, b) \in F^* \), then \( a = b \). This can be done by induction on formulas.

• Prove that \( F^* \) is the unique function on PROP satisfying the recursion equations. This can be done by letting \( G : \text{PROP} \rightarrow A \) be function which satisfies the recursion relations, then show by induction that for all \( \phi \in \text{PROP} \), if \( G(\phi) = a \) then \( (\phi, a) \in F^* \).

This theorem tells us that giving a definition by recursion gives a unique function from PROP to a set. Often times we will denote a recursive definition of a function on PROP as follows:

\textbf{Example:} Define \( b(\phi) \) to be the number of brackets in \( \phi \) as follows:

- \( b(\phi) = 0 \) if \( \phi \) is atomic,
- \( b((\phi \Box \psi)) = b(\phi) + b(\psi) + 2 \),
- \( b((\neg \phi)) = b(\phi) + 2 \).

What we are implicitly doing is using \( H_{at}(\phi) = 0, H_{\Box}(\phi, n, m) = n + m + 2, \) and \( H_{\neg}(\phi, n) = n + 2 \).

Then \( b(\phi) = H_{at}(\phi) = 0 \) for \( \phi \) atomic, \( b((\phi \Box \psi)) = H_{\Box}(\phi, b(\phi), b(\psi)) = b(\phi) + b(\psi) + 2 \), and \( b((\neg \phi)) = H_{\neg}((\neg \phi), b(\phi)) = b(\phi) + 2 \).

The theorem tells us that this function \( b : \text{PROP} \rightarrow \mathbb{N} \) is well-defined and unique.

1.1.3 Subformulas

We will use definition by recursion to define a function \( \text{sub} : \text{PROP} \rightarrow \mathcal{P}(\text{PROP}) \) which gives us the set of subformulas of a proposition.

\textbf{Definition 4.} The set of subformulas \( \text{Sub}(\phi) \) is given by

- \( \text{Sub}(\phi) = \{ \phi \} \) for \( \phi \) atomic,
- \( \text{Sub}((\phi \Box \psi)) = \text{Sub}(\phi) \cup \text{Sub}(\psi) \cup \{ (\phi \Box \psi) \} \),
- \( \text{Sub}((\neg \phi)) = \text{Sub}(\phi) \cup \{ (\neg \phi) \} \).

We say that \( \psi \) is a subformula of \( \phi \) if \( \psi \in \text{Sub}(\phi) \).
Note that in this recursive definition, we are using $H_{at}(\phi) = \{\phi\}$, $H_{\square}(\phi, A, B) = A \cup B \cup \{\phi\}$, and $H_{\neg}(\phi, A) = A \cup \{\phi\}$. Note that $A, B \in \mathfrak{P}(PROP)$ (they are sets of propositions). Again, the theorem tells us that the function Sub exists and is unique.

Here are some facts about subformulas, we leave their proofs as an exercise to the reader:

- The relation “is a subformula of” is a transitive relation on PROP.
- If $\phi$ is a subformula of $\psi$, then $\phi$ occurs in every formation sequence of $\psi$.
- If $\phi$ occurs in a shortest formation sequence of $\psi$ then $\phi$ is a subformula of $\phi$. That is, if there exists a formation sequence of $\psi$ with a formation sequence containing $\phi$ such that if we remove $\phi$ from the sequence, it is no longer a formation sequence, then $\phi$ must be a subformula of $\psi$.
- A proposition with $n$ connectives has at most $2n + 1$ subformulas.

Notational convention/final note: Good news! We are rapidly approaching semantics, which means that we can stop being so picky about our use of parentheses. We will adopt the following conventions: $\land$ and $\lor$ bind more strongly than $\to$ and $\leftrightarrow$, and $\neg$ binds more strongly than any other connectives. When it is not ambiguous, we can leave of parentheses (for example, the outer most parentheses). We will use the convention that we associate from left to right for connectives of the same strength.

Examples:

- $\neg\phi \lor \phi$ stands for $((\neg\phi) \lor \phi)$
- $\neg((\neg\neg\phi \land \bot)$ stands for $((\neg((\neg\neg\phi)) \land \bot)$
- $\phi \to \phi \lor (\psi \to \chi)$ stands for $(\phi \to (\phi \lor (\psi \to \chi)))$
- $p_0 \to p_1 \to p_2 \to p_3$ stands for $((p_0 \to p_1) \to p_2) \to p_3)$

Note that the convention of associating from left to right may not coincide with other conventions (for example, in some programming languages, implication associates from right to left). In general, we will still include parentheses to avoid confusion in cases like this.

Warning: by the strictest definition of PROP, these abbreviations are not in PROP. They are, however, much easier for our human brains to read.

1.2 Semantics

Now we get to apply some meaning to our strings of symbols. We will take advantage of our rigid structure to assign truth values to the propositions. The propositional variables will stand for sentences which are either true or false, such as “magic is cool” (true), and use the inductive characterization of PROP to assign truth values to propositions with connectives.

For ease of notation, we will use 1 and 0 to mean ‘true’ and ‘false’. For a proposition $\phi$, we will write $v(\phi) = 1$ if $\phi$ is true, and $v(\phi) = 0$ if $\phi$ is false.

We can illustrate the effect of each connective on truth values using truth tables.
Given $\phi$ and $\psi$, we determine $v((\phi \square \psi))$ by looking at the appropriate table and finding the entry at row $v(\phi)$ and column $v(\psi)$, and analogously for $\neg$.

Lastly, we define $v(\bot) = 0$, $\bot$ is a proposition which is always false. We can also include a symbol for truth, $\top$, and define $v(\top) = 1$.

We can collect all of this information succinctly as follows:

**Definition 5.** A mapping $[\cdot] : PROP \rightarrow \{0, 1\}$ is a **valuation** if

- $[\phi \land \psi] = \min([\phi], [\psi])$
- $[\phi \lor \psi] = \max([\phi], [\psi])$
- $[\phi \rightarrow \psi] = 0 \iff [\phi] = 1$ and $[\psi] = 0$
- $[\phi \leftrightarrow \psi] = 1 \iff [\phi] = [\psi]$
- $[\neg \phi] = 1 - [\phi]$
- $[\bot] = 0$

A valuation can be given only for atoms and then we can expand it to all of PROP recursively. We state this precisely as the following theorem:

**Theorem 6.** If $v$ is a mapping from the atoms into $\{0, 1\}$, satisfying $v(\bot) = 0$, then there exists a unique valuation $[\cdot]_v$, such that $[\phi]_v = v(\phi)$ for atomic $\phi$.

**Proof.** First, we will show existence by giving a recursive definition of $[\cdot]_v : PROP \rightarrow \{0, 1\}$.

- If $\phi$ is atomic, $[\phi]_v = v(\phi)$.
- $[\phi \land \psi]_v = \min([\phi]_v, [\psi]_v)$
- $[\phi \lor \psi]_v = \max([\phi]_v, [\psi]_v)$
- $[\phi \rightarrow \psi]_v = 0 \iff [\phi]_v = 1$ and $[\psi]_v = 0$
- $[\phi \leftrightarrow \psi]_v = 1 \iff [\phi]_v = [\psi]_v$
- $[\neg \phi]_v = 1 - [\phi]_v$
- $[\bot]_v = 0$
It is immediate from the definition that this is a valuation.

Uniqueness is a consequence of the following lemma.

**Lemma 7.** Let \( v, v' \) be mappings from the atoms to \( \{0, 1\} \) satisfying \( v(\bot) = v'(\bot) = 0 \). Let \( \phi \in PROP \) be given. If \( v(p_i) = v'(p_i) \) for all \( p_i \) occurring in \( \phi \), then \( \llbracket \phi \rrbracket_v = \llbracket \phi \rrbracket_{v'} \).

**Proof.** By induction on \( \phi \), left as an exercise to the reader.

Since a function \( v \) from the atoms to \( \{0, 1\} \) determines a unique valuation, we often refer to \( v \) itself as a valuation, or an atomic valuation. We will also use the notation \( \llbracket \cdot \rrbracket_v \) to denote the valuation associated with \( v \), and drop the subscript \( v \) when it is not ambiguous.

We are interested in elements of PROP which are always true, that is, which are true under any valuation.

**Definition 6.**
(i) \( \phi \) is a tautology if \( \llbracket \phi \rrbracket_v = 1 \) for all valuations \( v \).

(ii) \( \models \phi \) stands for “\( \phi \) is a tautology”.

(iii) Let \( \Gamma \) be a set of propositions, then \( \Gamma \models \phi \) if and only if for all valuations \( v \),

\[
(\llbracket \psi \rrbracket_v = 1 \text{ for all } \psi \in \Gamma) \Rightarrow \llbracket \phi \rrbracket_v = 1.
\]

That is, we say that \( \Gamma \models \phi \) if for any valuation such that every proposition in \( \Gamma \) is true, then \( \phi \) is necessarily true under that valuation.

Note that (ii) is a special case of (iii). If \( \Gamma \) is empty, then any valuation will (vacuously) make every proposition in \( \Gamma \) true, so for \( \emptyset \models \phi \) to hold, it must be the case that for every valuation \( \llbracket \cdot \rrbracket_v \), \( \llbracket \phi \rrbracket_v = 1 \).

We may use the shorthand \( \phi_1, \ldots, \phi_n \models \psi \) for \( \{\phi_1, \ldots, \phi_n\} \models \psi \).

Note that when we say \( v(\phi) = 1 \) for all (atomic valuations) \( v \), by the theorem it is equivalent to say that \( \llbracket \phi \rrbracket = 1 \) for every valuation \( \llbracket \cdot \rrbracket \).

**Examples:**

- \( \models \phi \rightarrow \phi \)
- \( \models \neg \neg \phi \rightarrow \phi \)
- \( \models \phi \lor \psi \leftrightarrow \psi \lor \phi \)
- \( \phi, \psi \models \phi \land \psi \)
- \( \phi, \phi \rightarrow \psi \models \psi \)
- \( \phi \rightarrow \psi, \neg \psi \models \neg \phi \)

In some of these cases, we can check that these hold directly by the definition. For example, given a valuation \( \llbracket \cdot \rrbracket_v \), such that \( \llbracket \phi \rrbracket_v = \llbracket \psi \rrbracket_v = 1 \), then by definition, \( \llbracket \phi \land \psi \rrbracket_v \), so \( \phi, \psi \models \phi \land \psi \).

We will see shortly that there are various methods for checking that a proposition is a tautology, but first it will be helpful to carefully state what it means to substitute a proposition into another proposition.

For each \( \phi \in PROP \) and each propositional variable \( p_i \), we define a function \( \phi[\cdot / p_i] : PROP \rightarrow PROP \) recursively as follows:
Definition 7. Let $\psi \in \textit{PROP}$.

- $\phi[\psi/p_i]$ is $\phi$ if $\phi$ is atomic and not equal to $p_i$, otherwise it is $\psi$ if $\phi = p_i$.
- $(\phi_1 \square \phi_2)[\psi/p_i] = \phi_1[\psi/p_i] \square \phi_2[\psi/p_i]$.
- $(\neg \phi)[\psi/p_i] = \neg(\phi[\psi/p_i])$.

Example: Suppose $\phi$ is $(p_0 \rightarrow p_1) \land (p_0 \lor p_2)$. Then $\phi[(p_0 \land p_1)/p_0]$ can be calculated as follows:

\[
\begin{align*}
\phi[(p_0 \land p_1)/p_0] &= (p_0 \rightarrow p_1)[((p_0 \land p_1)/p_0) \land (p_0 \lor p_2)]((p_0 \land p_1)/p_0) \\
&= (p_0[(p_0 \land p_1)/p_0] \rightarrow p_1[[p_0 \land p_1]/p_0]) \land (p_0[(p_0 \land p_1)/p_0] \lor p_2[[p_0 \land p_1]/p_0]) \\
&= ((p_0 \land p_1) \rightarrow p_1) \land ((p_0 \land p_1) \lor p_2)
\end{align*}
\]

We will show that replacing parts of propositions with equivalent parts will preserve the valuation of a proposition. First we need the following lemmas.

Lemma 8. $[\phi \rightarrow \psi] = 1 \iff [\phi] \leq [\psi]$.

Proof. Left as an exercise to the reader. \hfill \Box

Lemma 9. $[\phi_1 \leftrightarrow \phi_2] \leq [\psi[\phi_1/p] \leftrightarrow \psi[\phi_2/p]]$ and $\models (\phi_1 \leftrightarrow \phi_2) \rightarrow (\psi[\phi_1/p] \leftrightarrow \psi[\phi_2/p])$ (where $p = p_i$ for some $i \in \mathbb{N}$).

Proof. By induction on $\psi$. This proof of the first part is left as an exercise to the reader.

For the second part, by the previous lemma, for any valuation

$[\models (\phi_1 \leftrightarrow \phi_2) \rightarrow (\psi[\phi_1/p] \leftrightarrow \psi[\phi_2/p]) = 1$, so $\models (\phi_1 \leftrightarrow \phi_2) \rightarrow (\psi[\phi_1/p] \leftrightarrow \psi[\phi_2/p])$. \hfill \Box

Theorem 10 (Substitution Theorem). If $\models \phi_1 \leftrightarrow \phi_2$, then $\models \psi[\phi_1/p] \leftrightarrow \psi[\phi_2/p]$ where $p$ is $p_i$ for some $i \in \mathbb{N}$.

Proof. This follows immediately from the lemma: if $\models \phi_1 \leftrightarrow \phi_2$, then since $\models (\phi_1 \leftrightarrow \phi_2) \rightarrow (\psi[\phi_1/p] \leftrightarrow \psi[\phi_2/p])$, for any valuation, $\phi_1 \leftrightarrow \phi_2$, so we must have $\psi[\phi_1/p] \leftrightarrow \psi[\phi_2/p]$. That is, $\models \psi[\phi_1/p] \leftrightarrow \psi[\phi_2/p]$. \hfill \Box

Now we can use truth tables to show that tautologies are true. We will demonstrate this with the following example:

Show $\models (\phi \rightarrow \psi) \leftrightarrow (\neg \psi \rightarrow \neg \phi)$.

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\phi & \psi & \neg \phi & \neg \psi & \phi \rightarrow \psi & \neg \phi \rightarrow \neg \psi & (\phi \rightarrow \psi) \leftrightarrow (\neg \psi \rightarrow \neg \phi) \\
\hline
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 \\
\hline
\end{array}
\]

The idea is that we use something analogous to a formation sequence, treating $\phi$ and $\psi$ as atomic, and then give every possible combination of valuations of $\phi$ and $\psi$ (in this case there are $2^2$ possibilities. (By the theorem, if we consider the original proposition with atoms in the place of $\phi$ and $\psi$, we can get to the result by starting with a formation sequence and
then using substitution). Then we use the definitions of the connectives to determine the valuations of each of the subsequent propositions. If the final column is all 1’s, then we know that for any valuation, the proposition is true, so it is a tautology.

This can be shortened as follows:

Just write the proposition across the top of the truth table, then fill in the possible valuations of $\phi$ and $\psi$, and as in the sequence in the above table, fill in the rest.

\[
\begin{array}{cc}
(\phi \rightarrow \psi) & (\neg \psi \rightarrow \neg \phi) \\
0 & 0 \\
0 & 1 \\
1 & 0 \\
1 & 1 \\
\end{array}
\]

Next, $\neg$ binds the most strongly, so we get

\[
\begin{array}{cc}
(\phi \rightarrow \psi) & (\neg \psi \rightarrow \neg \phi) \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
\end{array}
\]

After that, we calculate the contents of each part in parentheses, and put the result below the respective connectives.

\[
\begin{array}{cc}
(\phi \rightarrow \psi) & (\neg \psi \rightarrow \neg \phi) \\
0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
\end{array}
\]

Finally, we use the values for each section in parentheses to determine the valuation of the entire proposition, which goes under the $\leftrightarrow$.

\[
\begin{array}{cc}
(\phi \rightarrow \psi) & (\neg \psi \rightarrow \neg \phi) \\
0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
\end{array}
\]

Since they are all 1’s, we know that this proposition is a tautology.

We see that writing out a truth table every single time we want to show a proposition is a tautology is a cumbersome task. Next we will see several tautologies and examine how to use them to prove facts about other propositions.

The first observation, made by Boole, is that PROP with our connectives satisfy a number of nice structural properties. In fact, these properties have been distilled down and applied to other structures, which we call Boolean Algebras.
Theorem 11. The following propositions are tautologies:

**Associativity:**

\[ (\phi \lor \psi) \lor \sigma \leftrightarrow \phi \lor (\psi \lor \sigma) \]
\[ (\phi \land \psi) \land \sigma \leftrightarrow \phi \land (\psi \land \sigma) \]

**Commutativity:**

\[ \phi \lor \psi \leftrightarrow \psi \lor \phi \]
\[ \phi \land \psi \leftrightarrow \psi \land \phi \]

**Distributivity:**

\[ \phi \lor (\psi \land \sigma) \leftrightarrow (\phi \lor \psi) \land (\phi \lor \sigma) \]
\[ \phi \land (\psi \lor \sigma) \leftrightarrow (\phi \land \psi) \lor (\phi \land \sigma) \]

**DeMorgan’s Laws:**

\[ \neg(\phi \lor \psi) \leftrightarrow \neg\phi \land \neg\psi \]
\[ \neg(\phi \land \psi) \leftrightarrow \neg\phi \lor \neg\psi \]

**Idempotency:**

\[ \phi \lor \phi \leftrightarrow \phi \]
\[ \phi \land \phi \leftrightarrow \phi \]

**Double Negation Law:**

\[ \neg\neg\phi \leftrightarrow \phi \]

Proof. We can prove all of these via truth tables, or by a computation. For example, we can show commutativity for \( \land \) as follows: given a valuation \( J \).

\[ J[\phi \land \psi] = \min(J[\phi], J[\psi]) = 1 \iff J[\phi] = 1 \text{ and } J[\psi] = 1 \]
\[ \iff J[\psi] = 1 \text{ and } J[\phi] = 1 \iff \min(J[\psi], J[\phi]) = 1 \iff J[\psi \land \phi] = 1. \]

The rest are left as an exercise to the reader.

Any structure where we have an underlying set and interpretation of the symbols \( \land, \lor, \neg, \bot, \top \) satisfying these properties is a Boolean Algebra. This occurs in many places other than PROP, for example, given a set \( A \), if we let \( \mathcal{P}(A) \) be the underlying set and interpret the symbols as \( \cap, \cup, ^c, \emptyset, A \), this forms a Boolean Algebra.

We need a few more tools before we will be able to apply these facts to logical arguments.

**Lemma 12.** If \( \vdash \phi \rightarrow \psi \), then \( \vdash \phi \land \psi \leftrightarrow \phi \) and \( \vdash \phi \lor \psi \leftrightarrow \psi \).

Proof. By Lemma 8, \( \vdash \phi \rightarrow \psi \) implies that for any valuation \( [\phi] \leq [\psi] \).

So \( [\phi \land \psi] = \min([\phi], [\psi]) = [\phi] \), so \( \vdash \phi \land \psi \leftrightarrow \phi \).

Similarly, \( [\phi \lor \psi] = \max([\phi], [\psi]) = [\psi] \), so \( \vdash \phi \lor \psi \leftrightarrow \psi \).
Lemma 13. The following hold for any propositions \( \phi \) and \( \psi \):

(a) \( \vdash \phi \Rightarrow \vdash \phi \land \psi \leftrightarrow \psi \)

(b) \( \vdash \phi \Rightarrow \vdash \neg \phi \lor \psi \leftrightarrow \psi \)

(c) \( \vdash \bot \lor \psi \leftrightarrow \psi \)

(d) \( \vdash \top \land \psi \leftrightarrow \psi \)

Proof. Left as an exercise to the reader.

Sometimes it is advantageous to think about why these tautologies make sense. For example, in (c), \( \bot \lor \psi \) requires at least one of \( \bot \) or \( \psi \) to be true. Since \( \bot \) is never true, this means \( \psi \) must be true.

Theorem 14. The following are tautologies.

(a) \( \vdash (\phi \leftrightarrow \psi) \leftrightarrow (\phi \rightarrow \psi) \land (\psi \rightarrow \phi) \)

(b) \( \vdash (\phi \rightarrow \psi) \leftrightarrow (\neg \phi \lor \psi) \)

(c) \( \vdash \phi \lor \psi \leftrightarrow (\neg \phi \rightarrow \psi) \)

(d) \( \vdash \phi \land \psi \leftrightarrow \neg (\neg \phi \land \neg \psi) \)

(e) \( \vdash \phi \land \psi \leftrightarrow \neg (\neg \phi \lor \neg \psi) \)

(f) \( \vdash \neg \phi \leftrightarrow (\phi \rightarrow \bot) \)

(g) \( \vdash \bot \leftrightarrow \phi \land \neg \phi \)

Proof. The proof is by calculating the truth values of each.

For example, for (a),

<table>
<thead>
<tr>
<th>(\phi \leftrightarrow \psi)</th>
<th>\leftrightarrow</th>
<th>(\phi \rightarrow \psi)</th>
<th>\land</th>
<th>(\psi \rightarrow \phi)</th>
<th>\phi</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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Alternatively, we can show these by letting \([\cdot]\) be any valuation.

For example, for (g), \([\phi \land \neg \phi] = \min([\phi], [\neg \phi]) = \min([\phi], 1 - [\phi]) = 0\) since either \([\phi] = 0\), or \([\phi] = 1\), so \(1 - [\phi] = 0\). Thus, \([\phi \land \neg \phi] = 0 = [\bot]\), which means \([\phi \land \neg \phi \leftrightarrow \bot] = 1\).

The rest are left to the reader.

Observe that using the results of this theorem, we can show that \(\{\land, \lor, \rightarrow, \leftrightarrow, \neg\}\) is redundant, in that we can express some of these connectives in terms of others. For example, by (a), we can replace \( \phi \leftrightarrow \psi \) by \( (\phi \rightarrow \psi) \land (\psi \rightarrow \phi) \). In fact, as we will see in the exercises, we can use \(\{\lor, \neg\}\) to define any \(n\)-ary connective given by its truth table. This is also true of \(\{\rightarrow, \neg\}\), \(\{\land, \neg\}\), and \(\{\rightarrow, \bot\}\). In some ways this justifies our selection of connectives. The idea is that we can express arbitrary connectives using these connectives.

Now we can define logical equivalence of propositions.
Definition 8. \( \phi \) and \( \psi \) in PROP are logically equivalent if \( \models \phi \iff \psi \). We write \( \phi \equiv \psi \).

Lemma 15. \( \equiv \) is an equivalence relation on PROP.

- Reflexive: \( \phi \equiv \phi \)
- Symmetric: \( \phi \equiv \psi \Rightarrow \psi \equiv \phi \)
- Transitive: \( \phi \equiv \psi \) and \( \psi \equiv \chi \Rightarrow \phi \equiv \chi \)

Proof. Using the fact that \( \phi \equiv \psi \iff \models \phi \iff \psi \) for all valuations, and the fact that equality is reflexive, symmetric, and transitive. \( \square \)

Instead of relying only on truth tables, now we can use algebraic computations to show that two propositions are logically equivalent.

Example: Show \( \models [\phi \to (\psi \to \sigma)] \iff [\phi \land \psi \to \sigma] \).

\[
\begin{align*}
\phi \to (\psi \to \sigma) & \equiv \neg \phi \lor (\psi \to \sigma) \text{ Theorem 14(b)} \\
& \equiv \neg \phi \lor (\neg \psi \lor \sigma) \text{ Theorem 14(b) and substitution} \\
& \equiv (\neg \phi \lor \neg \psi) \lor \sigma \text{ Associativity} \\
& \equiv (\neg \phi \land \psi) \lor \sigma \text{ DeMorgan’s Law and substitution} \\
& \equiv (\phi \land \psi) \to \sigma \text{ Theorem 14(b)} \\
\end{align*}
\]

Thus, \( \phi \to (\psi \to \sigma) \equiv \phi \land \psi \to \sigma \).

Since \( \equiv \) is an equivalence relation, when determining a string of logical equivalences, it may be useful to manipulate both sides of the equivalence until we get the same thing on both sides and then link them together (just like solving algebraic equations).

Example: Show \( \models (\phi \to \psi) \iff (\neg \psi \to \neg \phi) \) with a string of logical equivalences (without citing each justification).

\[
\neg \psi \to \neg \phi \equiv \neg \neg \psi \lor \neg \phi \equiv \psi \lor \neg \phi \equiv \neg \phi \lor \psi \equiv \phi \to \psi
\]

The Boolean Algebra conditions in Theorem 11 and the tautologies in Theorem 14 are considered standard facts, and from now on these equivalences can be used without explicit citation.

Notice that by associativity, we can write \( \phi_1 \land \phi_2 \land \phi_3 \) in the place of \( (\phi_1 \land \phi_2) \land \phi_3 \) or \( \phi_1 \land (\phi_2 \land \phi_3) \), since these are logically equivalent. We introduce the following symbols, analogous to \( \sum \) and \( \prod \) in arithmetic, for ease of notation.

Definition 9. We define \( \bigwedge \) and \( \bigvee \) recursively on \( \mathbb{N} \) as follows:

- \( \bigwedge_{i \leq 0} \phi_i = \phi_0 \)
- \( \bigwedge_{i \leq n+1} \phi_i = \bigwedge_{i \leq n} \phi_i \land \phi_{n+1} \)

and

- \( \bigvee_{i \leq 0} \phi_i = \phi_0 \)
- \( \bigvee_{i \leq n+1} \phi_i = \bigvee_{i \leq n} \phi_i \lor \phi_{n+1} \)
The idea is that $\bigwedge_{i \leq n} \phi_i = \phi_0 \land \ldots \land \phi_n$ and $\bigvee_{i \leq n} \phi_i = \phi_0 \lor \ldots \lor \phi_n$.

**Definition 10.** If $\phi = \bigwedge_{i \leq n} \bigvee_{j \leq m} \phi_{ij}$, where $\phi_{ij}$ is atomic or the negation of an atom, then $\phi$ is in **conjunctive normal form**. If $\phi = \bigvee_{i \leq n} \bigwedge_{j \leq m} \phi_{ij}$ where $\phi_{ij}$ is atomic or the negation of an atom, then $\phi$ is in **disjunctive normal form**.

**Example:** $(p_0 \lor p_1 \lor p_2) \land (p_0 \lor \lnot p_1 \lor p_2) \land (\lnot p_0 \lor p_1 \lor \lnot p_2)$ is in conjunctive normal form. Notice that all of the same atoms appear in each of the conjuncts.

$(p_2 \land p_3) \lor (\lnot p_2 \land \lnot p_3)$ is in disjunctive normal form, and is equivalent to $(p_2 \leftrightarrow p_3)$.

**Theorem 16.** For every proposition $\phi$ there is a conjunctive normal form $\phi^\land$ and a disjunctive normal form $\phi^\lor$ such that $\vdash \phi \leftrightarrow \phi^\land$ and $\vdash \phi \leftrightarrow \phi^\lor$.

Proof. We will prove the following stronger fact: If $\phi$ is a proposition and the atoms of $\phi$ are among $\{p_0, \ldots, p_n, \bot\}$, then there exist $\phi^\land$ and $\phi^\lor$ both logically equivalent to $\phi$ such that

- $\phi^\land$ of the form $\bigwedge_{1 \leq i \leq k} \phi_i^\land$ where $\phi_i^\land$ is of the form $\overline{p}_0 \lor \ldots \lor \overline{p}_n \lor \bot$, $\overline{p}_i \in \{p_i, \lnot p_i\}$, and $\bot \in \{\bot, \lnot \bot\}$.
- $\phi^\lor$ of the form $\bigvee_{1 \leq i \leq j} \phi_i^\lor$ where $\phi_i^\lor$ is of the form $\overline{p}_0 \land \ldots \land \overline{p}_n$, $\overline{p}_i \in \{p_i, \lnot p_i\}$, and $\bot \in \{\bot, \lnot \bot\}$.

Note that $\phi^\land$ is in conjunctive normal form and $\phi^\lor$ is in disjunctive normal form, so the theorem will follow from this.

Let $\phi$ be a proposition whose atoms are among $\{p_0, \ldots, p_n, \bot\}$. Observe that given any atomic valuation $v$, there are $2^{n+1}$ different possibilities for $(v(p_0), \ldots, v(p_n))$. Further note that by Lemma 7, given a valuation $[\cdot]$, $[\phi]$ is determined by $[p_0], \ldots, [p_n]$.

Let $V \subset 2^{n+1}$ be such that if $v$ is atomic valuation with $(v(p_0), \ldots, v(p_n)) \in V$, then $[\phi]_v = 1$, and if $(v(p_0), \ldots, v(p_n)) \in 2^{n+1} \setminus V$, then $[\phi]_v = 0$.

Let $p_i^\bot$ denote $\lnot p_i$ and $p_i^1$ denote $p_i$.

If $V = \emptyset$, let $\phi^\lor = p_0 \land \ldots \land p_n \land \bot$. Note that if $V$ is empty, then for any valuation $[\phi] = 0 = \min([p_0], \ldots, [p_n], [\bot]) = [p_0 \land \ldots \land p_n \land \bot] = [\phi^\lor]$, and this is of the required form.

Otherwise, let $\phi^\lor = \bigvee_{\sigma \in V} (p_0^\sigma \land \ldots \land p_n^\sigma \land \bot)$.

Then let an atomic valuation $v$ be given. $[\phi]_v = 1$ if and only if $\sigma = (v(p_0), \ldots, v(p_n)) \in V$, so $[\phi^\lor] = \max_{\sigma \in V} [p_0^\sigma \land \ldots \land p_n^\sigma \land \bot]_v \geq [p_0^\sigma \land \ldots \land p_n^\sigma \land \bot]_v = \min([p_0^\sigma], \ldots, [p_n^\sigma], 1 - [\bot]_v) = 1$. Thus, $[\phi^\lor] = 1$.

If $[\phi]_v = 0$, then for any $\sigma \in V$, there is $0 \leq j \leq n$ such that $[p_j^\sigma]_v = 0$. Thus, for every $\sigma \in V$, $[p_0^\sigma \land \ldots \land p_n^\sigma \land \bot]_v = \min([p_0^\sigma], \ldots, [p_n^\sigma], 1 - [\bot]_v) = 0$. Hence, $[\phi^\lor] = \max_{\sigma \in V} [p_0^\sigma \land \ldots \land p_n^\sigma \land \bot]_v = 0$.

Hence, $[\phi] = [\phi^\lor]$.  

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Now suppose $2^{n+1} \setminus V = \emptyset$. In other words, every valuation makes $\phi$ true. Then let $\phi^\wedge = p_0 \lor \ldots \lor p_n \lor \bot$. Then for any valuation, $[\phi] = 1 = \max([p_0], \ldots, [p_n], [\bot]) = [\phi^\wedge]$, and this is of the required form.

Otherwise, let $\phi^\wedge$ be $\bigwedge_{\sigma \in 2^{n+1} \setminus V} p_0^{1-\sigma_0} \lor \ldots \lor p_n^{1-\sigma_n} \lor \bot$.

Let an atomic valuation $v$ be given. If $[\phi]_v = 0$, then $\sigma = (v(p_0), \ldots, v(p_n)) \in 2^{n+1} \setminus V$. So $[p_0^{1-\sigma_0} \lor \ldots \lor p_n^{1-\sigma_n} \lor \bot]_v = \max([p_0^{1-\sigma_0}]_v, \ldots, [p_n^{1-\sigma_n}]_v, [\bot]_v) = 0$. Thus, $[\phi^\wedge] = \min_{\sigma \in 2^{n+1} \setminus V} [p_0^{1-\sigma_0} \lor \ldots \lor p_n^{1-\sigma_n} \lor \bot]_v = 0$.

Finally, if $[\phi]_v = 1$, then let $\sigma \in 2^{n+1} \setminus V$ be given. Then we know that since $(v(p_0), \ldots, v(p_n)) \in V$, for some $j$, $\sigma_j \neq v(p_j)$, so $v(p_j) = 1 - \sigma_j$.

Then $[p_0^{1-\sigma_0} \lor \ldots \lor p_n^{1-\sigma_n} \lor \bot]_v = \max([p_0^{1-\sigma_0}]_v, \ldots, [p_n^{1-\sigma_n}]_v) = 1$ since $[p_j^{1-\sigma_j}]_v = 1$.

Hence, since this holds for all $\sigma \in 2^{n+1} \setminus V$, $[\phi^\wedge] = \min_{\sigma \in 2^{n+1} \setminus V} [p_0^{1-\sigma_0} \lor \ldots \lor p_n^{1-\sigma_n} \lor \bot]_v = 1$.

Hence, $[\phi]_v = [\phi^\wedge]_v$, so they are logically equivalent.

\[\square\]

**Observations:**

- It seems like overkill to be so careful about specifying which atoms appear in our normal forms. There various applications in which this is useful, but most notably it gives us a way to determine the truth value of a proposition in disjunctive normal form almost immediately given a valuation on atoms: if there is a conjunction $\neg p_0 \land \ldots \land \neg p_n$ such that $\neg p_i = p_i$ when the valuation of $p_i$ is 1 and $\neg p_i$ when the valuation is 0, then the valuation of the entire statement will be 1.

Notice that in this proof, we require the stronger form so that when we apply the inductive hypothesis, when the connective matches the form ($\land$ for CNF and $\lor$ for DNF), we get the result immediately.

- This can be proved by induction, but it winds up being overkill.

- This theorem tells us that in some sense, we only need connectives $\lor, \land, \neg$, since we can express any proposition in terms of these three connectives. As mentioned, we can actually use just $\lor, \neg$ or just $\land, \neg$, but using all three allows us these normal forms, which is convenient.

- This proof does not present us with useful method for finding the disjunctive or conjunctive normal forms of a given proposition, it only proves that it is possible.

- We will see in the next example that it is advantageous to require that all of the atoms appear in each conjunct (respectively, disjunct), as it displays some information about which valuations will make the proposition true.

We will demonstrate a method of finding the disjunctive and conjunctive normal forms of a given proposition with the following example (and note that this method can be generalized to provide an alternative proof of the theorem).

**Example:** Find the disjunctive normal form and conjunctive normal form of the following proposition

$$(\phi \rightarrow \psi) \rightarrow (\neg \phi \land \psi)$$
Note that for two propositions to be equivalent, they must have the same truth table.

<table>
<thead>
<tr>
<th>$(\phi \to \psi)$</th>
<th>$\to$</th>
<th>$(\neg \phi \land \psi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
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<td>0</td>
</tr>
</tbody>
</table>

The truth table tells us that for any valuation $[\cdot]$, $\begin{aligned} [[\phi \to \psi] \to (\neg \phi \land \psi)] &= 1 \\ \iff ([\phi] = 0 \text{ and } [\psi] = 1) \text{ or } ([\phi] = 1 \text{ and } [\psi] = 0) \end{aligned}$ (that is, the valuation gives us the second or third line of the truth table)

$\begin{aligned} 
\iff ([\neg \phi] = 1 \text{ and } [\psi] = 1) \text{ or } ([\phi] = 1 \text{ and } [\neg \psi] = 1) \\
\iff ([\neg \phi \land \psi] = 1) \text{ or } ([\phi \land \neg \psi] = 1) \\
\iff ([\neg \phi \land \psi] \lor (\phi \land \neg \psi)] = 1.
\end{aligned}$

That is, $(\phi \to \psi) \to (\neg \phi \land \psi) \equiv (\neg \phi \land \psi) \lor (\phi \land \neg \psi)$, and the latter is in disjunctive normal form.

On the other hand, for a valuation $[\cdot]$,

$\begin{aligned} [[\phi \to \psi] \to (\neg \phi \land \psi)] &= 0 \\ \iff ([\phi] = 0 \text{ and } [\psi] = 0) \text{ or } ([\phi] = 1 \text{ and } [\psi] = 1) \end{aligned}$ (that is, the valuation gives us the first or fourth line of the truth table)

$\begin{aligned} 
\iff [[(\neg \phi \land \neg \psi) \lor (\phi \land \neg \psi)] = 1 \text{ (by the same reasoning as in the DNF case)} \\
\text{So } (\phi \to \psi) \to (\neg \phi \land \psi) \equiv \neg((\neg \phi \land \neg \psi) \lor (\phi \land \psi)), \text{ since they have opposite truth values, and this is equivalent to } (\phi \lor \psi) \land (\neg \phi \lor \neg \psi) \text{ by applying DeMorgan’s laws and double negation. This is in conjunctive normal form.}
\end{aligned}$

In summary, to find the disjunctive normal form, use the valuations which make the proposition true and write those as conjunctions (these are sometimes called minterms by computer scientists), and then take the disjunction of those.

To find the conjunctive normal form, use the valuations which make the proposition false, write those as conjunctions, take the disjunction of those, and negate the entire thing. Then use DeMorgan’s laws to put it in conjunctive normal form.

Everything we have learned so far about $\vdash$ are syntactic properties: we characterized logical equivalence in terms of valuations, which we define as functions which assign truth values and follow some strict structural rules. We relied on some semantic intuition (for example, about the “meanings” of connectives like “and” and “or”), though ultimately all of our work has been based in syntax. Next, we will approach truth from a semantic standpoint, relying on our understanding of provability.

### 1.3 Natural Deduction

In this section we will formalize the process of inference making, towards are goal of formalizing logical reasoning. Until now, our approach has been largely constructive. For example, in order to show that $[[\phi \lor \psi]] = 1$, we must show that $[[\phi]] = 1$ or $[[\psi]] = 1$. However, it may be the case that we do not know which is true, but just that at least one of them is. For example, we may not know the truth value of $\phi$, but we know that $[[\phi \lor (\neg \phi)]] = 1$, since
either φ is true or its negation is true, but to explain this we do not explicitly show which of \[\phi\] or \[\neg \phi\] is equal to 1.

We will restrict our attention of the set of connectives \{∧, →, ⊥\}, and use the other connectives as shorthand for the propositions using ∧, →, ⊥ in Theorem 14.

1.3.1 Derivation Rules

Derivations are of the form “from φ and φ → ψ, we can conclude ψ”. We express them with the following notation:

**Introduction Rules**

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\phi) (\psi) (\phi \land \psi) ((\land I))</td>
<td>(\phi \land \psi) (\phi) (\psi)</td>
</tr>
</tbody>
</table>

**Elimination Rules**

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\phi \land \psi) (\phi) (\psi) ((\land E))</td>
<td>(\phi \land \psi) (\phi \land \psi)</td>
</tr>
</tbody>
</table>

\[\phi\] \(I_n\) \(\phi \rightarrow \psi\) \((\rightarrow I)\)

\[\phi\] \(E_n\) \(\phi \rightarrow \psi\) \((\rightarrow E)\)

These rules dictate how we can introduce and eliminate ∧ and →. Note the square brackets around \(\phi\) in \((\rightarrow I)\). These indicate a cancelled hypothesis. The superscript \(n\) on \([\phi]\) and the subscript \(I_n\) indicate that this hypothesis is cancelled by this → introduction.

All of these rules should coincide with our existing understanding of the meaning of these connectives, for example, if we know \(\phi\) and we know \(\phi \rightarrow \psi\), then we can conclude \(\psi\). The things above the line are premises and the things below the line are conclusions. We have two rules for ⊥ as well:

\[\bot\] \(\neg \phi\) \((\bot)\)

\[\phi\] \(\bot\) \((RAA)\)

Note that we are using \(\neg \phi\) as shorthand for \(\phi \rightarrow \bot\). The first one indicates that \(\bot\) implies anything. RAA stands for “reductio ad absurdum”, which is how proof by contradiction works. That is, if one assumes \(\neg \phi\) and shows \(\bot\), then we must have \(\phi\) (since \(\neg \phi\) is false).

*Natural Deduction* is the process of stringing these rules together to arrive at a desired conclusion. We will begin with a few examples of derivations before giving a definition.

*Example 1:*
\[
\frac{[\phi \land \psi]^1}{\psi} (\land E) \quad \frac{[\phi \land \psi]^1}{\phi} (\land E) \\
\frac{}{\psi \land \phi} (\land I) \\
\frac{}{\phi \land \psi \rightarrow \psi \land \phi} (\rightarrow I_1)
\]

\text{Example 2:}
\[
\frac{[\phi]^2 \quad [\phi \rightarrow \bot]^1}{(\rightarrow E)} \\
\frac{}{\bot} (\rightarrow I_1) \\
\frac{(\phi \rightarrow \bot) \rightarrow \bot}{(\phi \rightarrow \bot) \rightarrow \bot} (\rightarrow I_2)
\]

Note that we can rewrite Example 2 using the shorthand \(\neg \phi\) to mean \(\phi \rightarrow \bot\) as follows:
\[
\frac{[\phi]^2 \quad [-\phi]^1}{(\rightarrow E)} \\
\frac{}{\bot} (\rightarrow I_1) \\
\frac{\neg \neg \phi}{\neg \neg \phi} (\rightarrow I_2)
\]

It is often advantageous to work backwards when trying to find the derivation of a given proposition. For example, suppose we want to find a derivaiton of \(\vdash (\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \sigma) \rightarrow (\phi \rightarrow \sigma))\). We know that \((\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \sigma) \rightarrow (\phi \rightarrow \sigma))\) must be the bottom line of the derivation, which means that the last rule used must be \((\rightarrow I)\), and immediately above that line, we must have \((\psi \rightarrow \sigma) \rightarrow (\phi \rightarrow \sigma)\) with a cancelled hypothesis of \(\phi \rightarrow \psi\) somewhere else in the derivation. In other words, we start with this:
\[
\frac{}{(\psi \rightarrow \sigma) \rightarrow (\phi \rightarrow \sigma) \rightarrow (\phi \rightarrow \sigma)} (\rightarrow I_1)
\]

and we will have (possibly multiple instances of) a cancelled hypotheses \([\phi]^1\). By the same reasoning, we continue breaking this down at each instance of \((\rightarrow I)\) to get
\[
\frac{}{\sigma} (\rightarrow I_3) \\
\frac{}{\phi \rightarrow \sigma} (\rightarrow I_2) \\
\frac{(\psi \rightarrow \sigma) \rightarrow (\phi \rightarrow \sigma)}{(\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \sigma) \rightarrow (\phi \rightarrow \sigma))} (\rightarrow I_1)
\]
with possibly cancelled hypotheses $[\phi \rightarrow \psi]$, $[\psi \rightarrow \sigma]$ and $[\phi]^3$. Now we want to use these cancelled hypotheses to get $\sigma$. We can do this with two applications of ($\rightarrow E$).

$$
\frac{[\phi]^3 \quad [\phi \rightarrow \psi]^1}{\psi} \quad \frac{[\psi \rightarrow \sigma]^2}{\rightarrow E} \\
\frac{\sigma}{\rightarrow I_3} \\
\frac{\phi \rightarrow \sigma}{\rightarrow I_2} \\
\frac{(\psi \rightarrow \sigma) \rightarrow (\phi \rightarrow \sigma)}{(\psi \rightarrow \sigma) \rightarrow (\psi \rightarrow \sigma) \rightarrow (\phi \rightarrow \sigma)} \quad \rightarrow I_1
$$

Thus, we have a derivation with conclusion $(\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \sigma) \rightarrow (\phi \rightarrow \sigma))$ in which all of the hypotheses are cancelled.

Now we can define derivations in general, but first we need some notation. We use $D_{\phi}$ to denote a derivation with conclusion $\phi$. If we can use some of the introduction or elimination rules to conclude $\phi$ from $\psi$, we write $D_{\psi}$, or if $D_{\phi}$ and $D'_{\phi'}$ are derivations with $\phi$ and $\phi'$ respectively, and from $\phi$ and $\phi'$, we can conclude $\psi$ with one of the rules, we write

$$
\frac{D_{\phi} \quad D'_{\phi'}}{\phi \quad \phi'} \quad \frac{\psi}{\rightarrow I_1}
$$

We can define the set of all derivations as follows:

**Definition 11.** The set of derivations is the smallest set $X$ such that

1. $\phi \in X$ for all $\phi \in PROP$.

2. If $D_{\phi}$ and $D'_{\phi'}$ are in $X$, then $D_{\phi \land \phi'}$ is in $X$.

3. If $D_{\phi \land \psi}$ is in $X$, then $D_{\phi \land \psi}$ and $D_{\phi \land \psi}$ are in $X$.

4. If $D_{\phi}$ is in $X$, then $D_{\phi}$ is in $X$. 

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If \( \mathcal{D}_\phi \) and \( \mathcal{D}'_{\phi \rightarrow \psi} \) are in \( X \), then \( \mathcal{D} \mathcal{D}' \) is in \( X \).

(2\( \bot \)) If \( \mathcal{D}_\bot \) is in \( X \), then \( \bot \) is in \( X \).

\[
\begin{array}{c}
\mathcal{D} \\
\phi \\
\neg \phi \\
\mathcal{D}_\bot \\
\phi
\end{array}
\]

If \( \mathcal{D} \) is in \( X \), then \( \bot \) is in \( X \).

Note that since the class of derivations is inductively defined, we can mimic the results of the first section. For example, we have a principle of induction on the set of derivations: If \( A(\mathcal{D}) \) holds for the one element derivations, and \( A \) is preserved under the clauses in (2\( \land \)), (2\( \rightarrow \)), and (2\( \bot \)), then \( A(\mathcal{D}) \) holds for all derivations. Likewise, we can define mappings on the set of all derivations via recursion.

One should note the following slight abuse of notation in the cases of (\( \rightarrow I \)) and (RAA).

When we write \( \mathcal{D} \), it could be the case that there are multiple instances of \( \phi \) which do not occur as a result of the application of rules, and all of these can be cancelled.

In Example 1, we saw two instances of \( \phi \land \psi \) which were both cancelled in one application of (\( \rightarrow I \)), however no instances \( \phi \land \psi \) which result as the application of a rule should be cancelled.

The set of hypotheses of a derivation are the propositions in a derivation which do not occur immediately below a line (that is, are not the result of the application of one of the rules). We can define the set of hypotheses as a recursive function on the set of derivations, which we will leave as an exercise to the reader. We will however give the following example.

**Example:** Give a recursive definition of a function \( UCH \) which assigns to each derivation \( \mathcal{D} \) the set of uncanceled hypothesis \( UCH(\mathcal{D}) \subset PROP \).

1. For any proposition \( \phi \), if \( \mathcal{D} \) is the single line derivation \( \phi \), then \( UCH(\mathcal{D}) = \{ \phi \} \).

\[
(2\land) \quad UCH( \begin{array}{c} \mathcal{D} \\ \phi \end{array} ) = UCH( \begin{array}{c} \mathcal{D}' \\ \phi' \end{array} ).
\]

\[
UCH( \begin{array}{c} \mathcal{D} \\ \phi \land \psi \end{array} ) = UCH( \begin{array}{c} \mathcal{D} \\ \phi \land \psi \end{array} )
\]

22
\[
UCH(D \phi \land \psi) = UCH(\phi \land \psi)
\]
\[
UCH(D \phi \land \psi) = UCH(\phi \land \psi)
\]

(2→) \[
UCH(D \phi \rightarrow \psi) = UCH(D \phi \land UCH(D \phi \rightarrow \psi)
\]

(2⊥) \[
UCH(D \phi \rightarrow \psi) = UCH(D \phi \land UCH(D \phi \rightarrow \psi)
\]

Definition 12. Let \( \Gamma \) be a set of propositions and \( \phi \) be a proposition. We say that \( \Gamma \vdash \phi \) if there is a derivation with conclusion \( \phi \) such that all (uncanceled) hypotheses are in \( \Gamma \).

We say that \( \phi \) is derivable from \( \Gamma \).

Note that by definition, \( \Gamma \) may contain propositions which are not used in the derivation of \( \phi \). If \( \Gamma = \emptyset \), we write \( \vdash \phi \) and say that \( \phi \) is a theorem.

Example: Let \( \Gamma = \{p_0, p_0 \rightarrow p_1, p_1 \rightarrow p_2\} \). We will show that \( \Gamma \vdash p_2 \).

\[
p_0 \quad p_0 \rightarrow p_1 \quad (\rightarrow E)
\]
\[
p_1 \quad p_1 \rightarrow p_2 \quad (\rightarrow E)
\]

This is a derivation with conclusion \( p_2 \) whose uncanceled hypotheses are all contained in \( \Gamma \). Thus, \( \Gamma \vdash p_2 \).

Lemma 17.  
(a) \( \phi \in \Gamma \Rightarrow \Gamma \vdash \phi \)
(b) \( \Gamma \vdash \phi, \Gamma' \vdash \psi \Rightarrow \Gamma \cup \Gamma' \vdash \phi \land \psi \)
(c) \( \Gamma \vdash \phi \land \psi \Rightarrow \Gamma \vdash \phi \) and \( \Gamma \vdash \psi \)
(d) \( \Gamma \cup \{\phi\} \vdash \psi \Rightarrow \Gamma \vdash \phi \rightarrow \psi \)
(e) \( \Gamma \vdash \phi, \Gamma' \vdash \phi \rightarrow \psi \Rightarrow \Gamma \cup \Gamma' \vdash \psi \)
\((f)\) \(\Gamma \vdash \bot \Rightarrow \Gamma \vdash \phi\)

\((g)\) \(\Gamma \cup \{\neg \phi\} \vdash \bot \Rightarrow \Gamma \vdash \bot\)

**Proof.** We will prove \((e)\), and leave the rest as an exercise.

Suppose \(\Gamma \vdash \phi\), and \(\Gamma' \vdash \phi \rightarrow \psi\). Then there is a derivation \(D\) whose uncanceled hypotheses are all in \(\Gamma\), and a derivation \(\phi \rightarrow \psi\) whose uncanceled hypotheses are all in \(\Gamma'\).

\[
\begin{array}{cccc}
D & D' \\
\phi & \phi \rightarrow \psi \\
\hline
\psi \\
\end{array}
\]

\((\rightarrow E)\) is a derivation with conclusion \(\psi\) whose uncanceled hypotheses are in \(\Gamma \cup \Gamma'\). Thus, \(\Gamma \cup \Gamma' \vdash \psi\).

\(\square\)

When it is convenient, we will adopt the shorthand \(\Gamma, \psi \vdash \phi\) to mean \(\Gamma \cup \{\psi\} \vdash \phi\), or write \(\psi \vdash \phi\) to mean \(\{\psi\} \vdash \phi\).

Now we list some theorems. We use \(\neg \phi\) as an abbreviation for \(\phi \rightarrow \bot\) and \(\phi \leftrightarrow \psi\) as and abbreviation for \((\phi \rightarrow \psi) \land (\psi \rightarrow \phi)\).

**Theorem 18.**

1. \(\vdash \phi \rightarrow (\psi \rightarrow \phi)\)

2. \(\vdash \phi \rightarrow (\neg \phi \rightarrow \psi)\)

3. \(\vdash (\phi \rightarrow \psi) \rightarrow [(\psi \rightarrow \sigma) \rightarrow (\phi \rightarrow \sigma)]\)

4. \(\vdash (\phi \rightarrow \psi) \leftrightarrow (\neg \psi \rightarrow \neg \phi)\)

5. \(\vdash \neg \neg \phi \leftrightarrow \phi\)

6. \(\vdash [\phi \rightarrow (\psi \rightarrow \sigma)] \leftrightarrow [\phi \land \psi \rightarrow \sigma]\)

7. \(\vdash \bot \leftrightarrow (\phi \land \neg \phi)\)

**Proof.** We will prove 2. and leave the rest as an exercise.

First, rewrite 2. as \(\vdash \phi \rightarrow ((\phi \rightarrow \bot) \rightarrow \psi)\).

\[
\begin{array}{cccc}
[\phi]^1 & [\phi \rightarrow \bot]^2 \\
\hline
\bot & (\rightarrow E) \\
\hline
\psi & (\bot) \\
\hline
(\phi \rightarrow \bot) & (\rightarrow I_2) \\
\hline
\phi \rightarrow ((\phi \rightarrow \bot) \rightarrow \psi) & (\rightarrow I_1)
\end{array}
\]

\(\square\)
1.4 Soundness and Completeness

Now we are ready to link together semantics and syntax with soundness and completeness. One way we can think of $\models$ is as kind of a “coincidental” truth. $\Gamma \models \phi$ tells us that as long as everything in $\Gamma$ is true, then $\phi$ is true (that is, the truth value of $\phi$ coincides with that of $\Gamma$). It does not however give us any method of justifying $\phi$ using the information provided in $\Gamma$. $\Gamma \vdash \phi$ tells us that using assumptions from $\Gamma$ we can prove $\phi$, which is demonstrated with a derivation. In this section we will show that $\Gamma \models \phi \iff \Gamma \vdash \phi$.

**Lemma 19** (Soundness). $\Gamma \vdash \phi \Rightarrow \Gamma \models \phi$

**Proof.** Suppose $\Gamma \vdash \phi$. Then there is a derivation $D$ with conclusion $\phi$ such that all uncanceled hypotheses are in $\Gamma$. We will show by induction on $D$ that if $\Gamma$ is such that all of the uncanceled hypothesis of $D$ are in $\Gamma$ and $D$ has conclusion $\phi$, then $\Gamma \models \phi$.

(1) For our base case, if $D$ is just the one line derivation $\phi$, then we must have $\phi \in \Gamma$ since $\phi$ is an uncanceled hypothesis. Then, given any valuation $\llbracket * \rrbracket$ such that for every $\psi \in \Gamma$, $\llbracket \psi \rrbracket = 1$, then $\llbracket \phi \rrbracket = 1$, so $\Gamma \models \phi$.

(2) $\land I$): Let $D_\phi$ and $D'_{\psi}$ be two derivation for which the claim holds.

Let $\Gamma$ be a set containing all of the uncanceled hypotheses of

\[
\begin{array}{c}
D \\
\phi \\
D' \\
\psi \\
\phi \land \psi
\end{array}
\]

Then $\Gamma$ contains the uncanceled hypotheses of $D_\phi$, so by IH, $\Gamma \models \phi$, and similarly, $\Gamma$ contains the uncanceled hypotheses of $D'_{\psi}$, so by IH, $\Gamma \models \psi$.

Let $\llbracket * \rrbracket$ be a valuation such that for every $\sigma \in \Gamma$, $\llbracket \sigma \rrbracket = 1$. Then $\llbracket \phi \rrbracket = 1$ and $\llbracket \psi \rrbracket = 1$. Thus, $\llbracket \phi \land \psi \rrbracket = \min(\llbracket \phi \rrbracket, \llbracket \psi \rrbracket) = 1$. Thus, $\Gamma \models \phi \land \psi$.

($\land E$): Suppose $D_{\phi \land \psi}$ is a derivation for which the claim holds. Let $\Gamma$ contain the uncanceled hypotheses of

\[
\begin{array}{c}
D \\
\phi \land \psi \\
\phi
\end{array}
\]

So $\Gamma$ contains the uncanceled hypotheses of $D_{\phi \land \psi}$, which means that by IH, $\Gamma \models \phi \land \psi$.

Let $\llbracket * \rrbracket$ be a valuation such that $\llbracket \sigma \rrbracket = 1$ for all $\sigma \in \Gamma$. Then $\llbracket \phi \land \psi \rrbracket = 1$, so $\min(\llbracket \phi \rrbracket, \llbracket \psi \rrbracket) = 1$. Thus, $\llbracket \phi \rrbracket = 1$, so $\Gamma \models \phi$.

By the same reasoning we get the result for

\[
\begin{array}{c}
D \\
\phi \land \psi \\
\psi
\end{array}
\]
(2→) ($\rightarrow I$): Let $D$ be a derivation for which the claim holds.

\[
\begin{array}{c}
\phi \\
\psi
\end{array}
\]

Let $\Gamma$ contain the uncanceled hypotheses of $\phi \rightarrow \psi$.

Then $\Gamma \cup \{\phi\}$ contains the uncanceled hypotheses of $D$, so by IH, $\Gamma \cup \{\phi\} \models \psi$.

Let $[\cdot]$ be a valuation such that $[\sigma] = 1$ for all $\sigma \in \Gamma$, and suppose $[\phi] = 1$. Then $[\psi] = 1$. That is, it is not the case that $[\phi] = 1$ and $[\psi] = 0$, so we must have $[\phi \rightarrow \psi] = 1$. Hence, $\Gamma \models \phi \rightarrow \psi$.

($\rightarrow E$): Let $D$ and $\phi \rightarrow \psi$ be derivations for which the claim holds.

\[
\begin{array}{c}
\phi \\
\phi \rightarrow \psi
\end{array}
\]

Then $\Gamma$ contains the uncanceled hypotheses of $D$ and $\phi \rightarrow \psi$, so by IH, $\Gamma \models \phi$ and $\Gamma \models \phi \rightarrow \psi$.

Let $[\cdot]$ be a valuation such that $[\sigma] = 1$ for all $\sigma \in \Gamma$. Then $[\phi] = 1$ and $[\phi \rightarrow \psi] = 1 \neq 0$. So we cannot have the case that $[\phi] = 1$ and $[\psi] = 0$, so we must have $[\psi] = 1$.

(2⊥) ($\perp$): Let $D$ be a derivation for which the claim holds.

\[
\begin{array}{c}
\perp
\end{array}
\]

Let $\Gamma$ contain all of the uncanceled hypotheses of $\perp$. Then $\Gamma$ contains all of the uncanceled hypotheses of $D$, so by IH, $\Gamma \models \perp$.

That is, for all valuations $[\cdot]$, if $[\sigma] = 1$ for all $\sigma \in \Gamma$, then $[\perp] = 1$. But there are no valuations such that $[\perp] = 1$, so there can be no valuations which are 1 for every proposition in $\Gamma$.

Thus, it is vacuously true that $\Gamma \models \phi$.

($\neg \phi$): Let $D$ be a derivation for which the claim holds.

\[
\begin{array}{c}
\neg \phi \\
\perp
\end{array}
\]
Let Γ contain all of the uncanceled hypotheses of \( D \). Then \( \Gamma \cup \{\neg \phi\} \) contains the uncanceled hypotheses of \( D \), so by IH, \( \Gamma \cup \{\neg \phi\} \models \bot \).

Let \([\cdot]\) be a valuation such that \([\sigma] = 1\) for all \( \sigma \in \Gamma \). Suppose \([\neg \phi] = 1\). Then, since \( \Gamma \cup \{\neg \phi\} \models \bot, [\bot] = 1 \). But this is impossible, so \([\neg \phi] = 0\). Thus, \(1 - [\phi] = 0\), so \([\phi] = 1\). Hence, \( \Gamma \vdash \phi \).

The contrapositive of this lemma tells us that \( \not\models \phi \Rightarrow \not\models \phi \), that is, if a proposition is not a tautology, then it is not a theorem.

**Examples:** \( \not\models p_0, \not\models (\phi \rightarrow \psi) \rightarrow (\phi \land \psi) \).

To check these, we only need to find valuations which make these false. For example, any atomic valuation with \( v(p_0) = 0 \) for the first one.

In the second one, there is a slight subtlety happening here: \( (\phi \rightarrow \psi) \rightarrow (\phi \land \psi) \) is not actually a proposition, it is a proposition schema. When we write \( \vdash (\phi \rightarrow \psi) \rightarrow (\phi \land \psi) \), we are saying that this is true for any propositions \( \phi \) and \( \psi \), so we only need specific instances of \( \phi \) and \( \psi \) for which this fails to show that it is false. In particular, we will let \( \phi = \psi = p_0 \) and consider an atomic valuation such that \( v(p_0) = 0 \). Then \( [p_0 \rightarrow p_0]_v = 1 \) and \( [p_0 \land p_0]_v = 0 \), so \( [(p_0 \rightarrow p_0) \rightarrow (p_0 \land p_0)]_v = 0 \). So since \( \not\models (p_0 \rightarrow p_0) \rightarrow (p_0 \land p_0), \not\models (p_0 \rightarrow p_0) \rightarrow (p_0 \land p_0) \).

Now we will develop the machinery necessary to prove the converse of Soundness.

**Definition 13.** A set \( \Gamma \) of propositions is **consistent** if \( \Gamma \not\models \bot \).

In other words, \( \Gamma \) is consistent as long as we cannot derive a contradiction from \( \Gamma \).

**Lemma 20.** The following are equivalent:

1. \( \Gamma \) is consistent.
2. There is no proposition \( \phi \) such that \( \Gamma \vdash \phi \) and \( \Gamma \vdash \neg \phi \).
3. There is at least one \( \phi \) such that \( \Gamma \not\models \phi \).

**Proof.** We will rephrase this with the negation of each of these. Say that \( \Gamma \) is **inconsistent** if \( \Gamma \models \bot \). We will show the following are equivalent:

1. \( \Gamma \) is inconsistent.
2. There exists a proposition \( \phi \) such that \( \Gamma \vdash \phi \) and \( \Gamma \vdash \neg \phi \).
3. \( \Gamma \vdash \phi \) for all \( \phi \).
(a) ⇒ (c): Let $\Gamma \vdash \bot$. Then there is a derivation $D$ with conclusion $\bot$ whose hypotheses are in $\Gamma$. By ($\bot$), we have the derivation $\underbrace{\bot}_{\phi}$ whose uncanceled hypotheses are in $\Gamma$, thus, $\Gamma \vdash \phi$.

(c) ⇒ (b): Immediate.

(b) ⇒ (a): Let $\phi$ be such that $\Gamma \vdash \phi$ and $\Gamma \vdash \neg \phi$. That is, $\Gamma \vdash \phi \rightarrow \bot$. Let $D_\phi$ and $D_{\phi \rightarrow \bot}$ be derivations whose uncanceled hypotheses are in $\Gamma$. Then

$$\underbrace{D \quad D'}_{\phi \quad \phi \rightarrow \bot}$$

is a derivation whose uncanceled hypotheses are in $\Gamma$, so $\Gamma \vdash \bot$. Hence, $\Gamma$ is inconsistent. □

Part (c) is telling us that inconsistent sets of propositions are of little mathematical interest, since if you have an inconsistency you can prove anything.

In practice, we often times try to prove the consistency of a set of rules by showing an example of a place in which they can all exist. We formalize that here as follows:

**Lemma 21.** If there is a valuation such that $[\psi] = 1$ for all $\psi \in \Gamma$, then $\Gamma$ is consistent.

**Proof.** Suppose $\Gamma \vdash \bot$. Then by soundness, $\Gamma \models \bot$, so for any valuation $[\cdot]$, if $[\psi] = 1$ for all $\psi \in \Gamma$, then $[\bot] = 1$. But every valuation must be 0 for $\bot$, so no such valuation can exist. □

This gives us a tool for checking if a given set of propositions is consistent.

**Examples:**

1. $\{p_0, \neg p_1, p_1 \rightarrow p_2\}$ is consistent, as seen with any valuation such that $v(p_0) = 1$ and $v(p_1) = 0$ (it does not matter what the valuation of $p_2$ is, as long as $v(p_1) = 0$, $[p_1 \rightarrow p_2]_v = 1$).

2. $\{p_0, p_1, \ldots\}$ is consistent, just choose the valuation which is 1 on all atoms other than $\bot$.

**Lemma 22.**

(a) $\Gamma \cup \{\neg \phi\}$ is inconsistent $\Rightarrow$ $\Gamma \vdash \phi$

(b) $\Gamma \cup \{\phi\}$ is inconsistent $\Rightarrow$ $\Gamma \vdash \neg \phi$

**Proof.** Uses (RAA) and ($\rightarrow I$), left as an exercise to the reader. □

**Definition 14.** A set $\Gamma$ is maximally consistent if

(a) $\Gamma$ is consistent, and

(b) $\Gamma \subset \Gamma'$ and $\Gamma'$ consistent $\Rightarrow$ $\Gamma = \Gamma'$.

Equivalently, we could replace (b) with either of the following statements:
If \( \Gamma \subseteq \Gamma' \), then \( \Gamma' \) is inconsistent.

If \( \phi \notin \Gamma \), then \( \Gamma \cup \{ \phi \} \) is inconsistent.

Considering maximally consistent sets of propositions will help us bridge the gap between valuations and derivability, as valuations are universal in some sense: they are defined for all propositions. As we will see in the coming lemmas, maximally consistent sets give us information about the derivability of all propositions.

**Example:** Let \([\cdot]\) be a valuation on PROP. Then \( \Gamma = \{ \phi | [\phi] = 1 \} \) is a maximally consistent set of propositions. It is consistent by Lemma 21, and given any \( \phi \notin \Gamma \), \([\phi] = 0 \), so \([\neg \phi] = 1 - 0 = 1 \), which means that \( \neg \phi \in \Gamma \). So if \( \Gamma' = \Gamma \cup \{ \phi \} \), then we would have \( \Gamma' \vdash \phi \) and \( \Gamma' \vdash \neg \phi \), which means that \( \Gamma' \) is inconsistent.

**Lemma 23.** Each consistent set \( \Gamma \) is contained in a maximally consistent set \( \Gamma^* \).

**Proof.** There are countably many propositions (finite strings of symbols from a countable set), so suppose we have a list \( \phi_0, \phi_1, \phi_2, \ldots \) of all of the propositions. We will define sets of propositions \( \Gamma_i \) such that \( \Gamma_i \subset \Gamma_{i+1} \) and show that \( \Gamma^* := \bigcup_i \Gamma_i \) is maximally consistent.

\[
\Gamma_0 = \Gamma \\
\Gamma_{n+1} = \Gamma_n \cup \{ \phi_n \} \text{ if } \Gamma_n \cup \{ \phi_n \} \text{ is consistent, otherwise } \Gamma_{n+1} = \Gamma_n.
\]

First note that \( \Gamma_n \) is consistent for all \( n \): \( \Gamma_0 \) is consistent by assumption, and if \( \Gamma_n \) is consistent, then in either case, \( \Gamma_{n+1} \) is consistent (so \( \Gamma_n \) is consistent for all \( n \in \mathbb{N} \) by induction on \( n \)).

Now suppose for the sake of contradiction that \( \Gamma^* \) is inconsistent. Then \( \Gamma^* \vdash \bot \). That is, there is a derivation \( D \) whose uncanceled hypotheses are in \( \Gamma^* \). In particular, there are finitely many possible uncanceled hypotheses \( \psi_1, \ldots, \psi_n \). For each \( 1 \leq i \leq n \), let \( n_i \) be such that \( \psi_i = \phi_{n_i} \), and let \( m = \max_{1 \leq i \leq n} n_i \). Then by construction, \( \psi_1, \ldots, \psi_n \in \Gamma_m \). So \( \Gamma_m \vdash \bot \). But this contradicts that \( \Gamma_m \) is consistent. Hence, \( \Gamma^* \) is consistent.

Finally, to see that \( \Gamma^* \) is maximally consistent, let \( \Delta \) be a consistent set of propositions with \( \Gamma^* \subset \Delta \). Suppose there is \( \psi \in \Delta \setminus \Gamma^* \). Let \( n \) be such that \( \psi = \phi_n \). Since \( \phi_n \notin \Gamma^* \), \( \phi_n \notin \Gamma_{n+1} \), which only happens when \( \Gamma_n \cup \{ \phi_n \} \) is inconsistent. Thus, since \( \Gamma_n \cup \{ \phi_n \} \vdash \bot \), \( \Gamma^* \cup \{ \phi_n \} \vdash \bot \), so \( \Delta \vdash \bot \). But this contradicts that \( \Delta \) is consistent, so no such \( \psi \) can exist. That is, \( \Gamma^* = \Delta \).

**Lemma 24.** If \( \Gamma \) is maximally consistent, then \( \Gamma \) is closed under derivability. That is, if \( \Gamma \vdash \phi \), then \( \phi \in \Gamma \).

**Proof.** Suppose \( \Gamma \vdash \phi \) and \( \phi \notin \Gamma \). Then \( \Gamma \cup \{ \phi \} \) must be inconsistent since \( \Gamma \) is maximally consistent. So by Lemma 22, \( \Gamma \vdash \neg \phi \). Thus, by Lemma 20, \( \Gamma \) is inconsistent. \( \Rightarrow \Leftarrow \).

**Lemma 25.** If \( \Gamma \) is maximally consistent, then

(a) for all \( \phi \), either \( \phi \in \Gamma \) or \( \neg \phi \in \Gamma \)

(b) for all \( \phi \) and \( \psi \), \( \phi \rightarrow \psi \in \Gamma \) if and only if \( (\phi \in \Gamma \Rightarrow \psi \in \Gamma) \).
Proof.

(a) We know not both $\phi \in \Gamma$ and $\neg \phi \in \Gamma$, since this would make $\Gamma$ inconsistent. Suppose $\Gamma \cup \{\phi\}$ is inconsistent. Then by Lemma 22, $\Gamma \vdash \neg \phi$, so by Lemma 24, $\neg \phi \in \Gamma$. Otherwise, if $\Gamma \cup \{\phi\}$ is consistent, since $\Gamma$ is maximal, we must have $\phi \in \Gamma$.

(b) First suppose $\phi \rightarrow \psi \in \Gamma$ and $\phi \in \Gamma$, so $\Gamma \vdash \phi \rightarrow \psi$ and $\Gamma \vdash \psi$. Then there is a derivation $\mathcal{D}$ with uncanceled hypotheses in $\Gamma$ and a derivation $\mathcal{D}'$ with uncanceled hypotheses in $\Gamma$. So $\mathcal{D}' \phi \rightarrow \psi$ has uncanceled hypotheses in $\Gamma$. Thus, $\Gamma \vdash \psi$, so by Lemma 24, $\psi \in \Gamma$.

Conversely, suppose $\phi \in \Gamma \Rightarrow \psi \in \Gamma$. If $\phi \in \Gamma$, then since $\psi \in \Gamma$, \[ [\phi] \uparrow [\psi] \downarrow \] has uncanceled hypotheses in $\Gamma$, so $\Gamma \vdash \psi$. If $\phi \notin \Gamma$, by part (a), $\neg \phi \in \Gamma$. So \[ [\phi] \uparrow \neg \phi \] has uncanceled hypotheses in $\Gamma$, so $\Gamma \vdash \neg \phi$.

Corollary 26. If $\Gamma$ is maximally consistent, then $\phi \in \Gamma$ if and only if $\neg \phi \notin \Gamma$, and $\neg \phi \in \Gamma$ if and only if $\phi \notin \Gamma$.

Lemma 27. If $\Gamma$ is consistent, then there exists a valuation such that $[\psi] = 1$ for all $\psi \in \Gamma$.

Proof. By Lemma 23, $\Gamma$ is contained in some $\Gamma^*$ which is maximally consistent.

For atoms $p_i$, define $v(p_i) = 1$ if and only if $p_i \in \Gamma^*$, and $v(\bot) = 0$. This can be extended to a valuation $[\cdot]_v$ on all of Prop. We will show by induction on formulas that $[\phi] = 1$ if and only if $\phi \in \Gamma^*$. Since $\Gamma^* \supseteq \Gamma$, this valuation will be as required.

If $\phi$ is atomic, then $[\phi]_v = v(\phi) = 1$ if and only if $\phi$ is of the form $p_i$ and $p_i \in \Gamma^*$ (since $\Gamma^*$ is consistent, we know $\bot \notin \Gamma^*$, and $v(\bot) = 0$).

Assume $\phi$ and $\psi$ are propositions for which the claim holds.

$[\phi \land \psi]_v = \min([\phi]_v, [\psi]_v) = 1$

$\Leftrightarrow [\phi]_v = 1$ and $[\psi]_v = 1$

$\Leftrightarrow \phi \in \Gamma^*$ and $\psi \in \Gamma^*$ by IH,

$\Leftrightarrow \phi \land \psi \in \Gamma^*$, by Lemma 17 parts (b) and (c), and Lemma 24.

$[\phi \rightarrow \psi]_v = 1$

$\Leftrightarrow [\phi]_v \leq [\psi]_v$,

$\Leftrightarrow [\phi]_v = 1$ $\Rightarrow$ $[\psi]_v = 1$

$\Leftrightarrow (\phi \in \Gamma \Rightarrow \psi \in \Gamma)$ by IH,

$\Leftrightarrow \phi \rightarrow \psi \in \Gamma$ by Lemma 25.
Now let $\phi \in \Gamma$ be given. We can rewrite $\phi$ using connectives from $\{\land, \rightarrow, \bot\}$ (as the remaining connectives are shorthand for expressions using these connectives). Thus, by our induction on formulas, $[\phi]_v = 1$. That is, $[\cdot]_v$ is as required.

**Corollary 28.** $\Gamma \not\vDash \phi$ if and only if there is a valuation such that $[\psi] = 1$ for all $\psi \in \Gamma$ and $[\phi] = 0$.

**Proof.** First note that $\Gamma \not\vDash \phi \iff \Gamma \cup \{\neg \phi\}$ is consistent. The forward direction (by contrapositive) is Lemma 22. For the converse, If $\Gamma \cup \{\neg \phi\}$ is consistent and $\Gamma \vdash \phi$, then there is a derivation $D$ with uncanceled hypotheses in $\Gamma$, so $\frac{\phi \quad \neg \phi}{\bot}$ ($\rightarrow E$) is a derivation of $\bot$ with uncanceled hypotheses is $\Gamma \cup \{\neg \phi\}$, contradicting that $\Gamma \cup \{\neg \phi\}$ is consistent.

$\Gamma \cup \{\neg \phi\}$ is consistent

$\iff$ there is a valuation such that $[\psi] = 1$ for all $\psi \in \Gamma \cup \{\neg \phi\}$ by Lemma 27 and soundness,

$\iff$ there is a valuation such that $[\psi] = 1$ for all $\psi \in \Gamma$ and $[\phi] = 0$. \qed

**Theorem 29** (Completeness Theorem). $\Gamma \vdash \phi \iff \Gamma \vDash \phi$.

**Proof.** The forward direction is Soundness.

For the converse, if $\Gamma \not\vDash \phi$, then by Corollary 28 there is a valuation which is 1 on everything in $\Gamma$ with $[\phi] = 0$. That is, $\Gamma \not\vDash \phi$. \qed

Later on, we will consider the following closely related notion to maximally consistent:

**Definition 15.** A set of propositions $\Gamma$ is **complete** if for all propositions $\phi$, $\Gamma \vdash \phi$ or $\Gamma \vdash \neg \phi$.

Note that it can be shown that if $\Gamma$ is consistent, then $Cons(\Gamma) = \{\sigma | \Gamma \vdash \sigma\}$ (the set of consequences of $\Gamma$) is maximally consistent if and only if $\Gamma$ is complete.
2 First Order Logic

Propositional logic has given us a reasonably robust framework in which to express (mathematical) statements. For example, the statement “All magicians are cool and Professor Noquez is a magician, therefore, Professor Noquez is cool,” is a statement of the form \( \phi \land \psi \rightarrow \sigma \) (and in this case, the valuation of \( \sigma \) happens to be 1), however the universal quantification in the statement “All magicians are cool” is hidden inside the proposition \( \phi \). We find that it will be to our advantage to formally handle the use of quantifiers.

We have seen many examples of mathematical statements with the universal quantifier \( \forall \) or the existential quantifier \( \exists \). For example, statements like \( \forall x (x \cdot 0 = 0) \), or \( \exists y (x + y = 1) \). Or more complicated examples like \( \forall \epsilon > 0 \exists \delta > 0 (|x| < \delta \rightarrow |f(x)| < \epsilon) \). Notice that in the first two examples we have not specified which sets we are quantifying over (are we saying \( x \cdot 0 = 0 \) for all natural numbers \( x \)? or real numbers? or complex numbers?). In the latter example, we have specified that \( \epsilon > 0 \) and \( \delta > 0 \), and implicit in this is that \( \epsilon \) and \( \delta \) are real numbers.

Going forward, we will be careful to specify what exactly it is that we can quantifier over in our language.

2.1 Languages and Structures

Definition 16. A language \( \mathcal{L} \) is given by specifying the following data:

(i) a set of function symbols \( \mathcal{F} \) and a positive integer \( n_f \) for each \( f \in \mathcal{F} \)

(ii) a set of relation symbols \( \mathcal{R} \) and a positive integer \( n_R \) for each \( R \in \mathcal{R} \)

(iii) a set of constant symbols \( \mathcal{C} \).

\( n_f \) tells us that \( f \) is a function with \( n_f \) variables and \( n_R \) tells us that \( R \) is an \( n_R \)-ary relation. We call these numbers the *arity* of the functions/relations.

Examples:

1. \( \mathcal{L} = \{+,-,\cdot,<,0,1\} \), where +,−,⋅ are all functions on 2 variables, < is a binary relation, and 0 and 1 are constants. This is the language of ordered rings. We will be able to use this language to describe structures like the real numbers and the integers.

2. \( \mathcal{L} = \{E\} \) where \( E \) is a binary relation is the language of graphs. \( E \) is going to be used to describe the edge relation.

3. \( \mathcal{L} = \emptyset \) is the language of pure sets.

4. \( \mathcal{L} = \{+,c,\cdot|c \in \mathbb{Q}\} \) where + is a function in two variable, \( c \cdot \) is a function in one variable for all \( c \in \mathbb{Q} \), and \( 0 \) is a constant. This is the language of vector spaces over \( \mathbb{Q} \). Note that this language is infinite, there is a function symbol for every element of \( \mathbb{Q} \).

Now we can use these languages to describe structures.

Definition 17. An \( \mathcal{L} \)-structure \( \mathcal{M} \) is given by the following data:

(i) A non-empty set \( M \), the *universe* or *underlying set* of \( \mathcal{M} \).
(ii) a function $f^M : M^n \rightarrow M$ for each $f \in F$

(iii) a set $R^M \subset M^m$ for each $R \in R$

(iv) an element $c^M \in M$ for each $c \in C$.

We refer to $f^M$, $R^M$ and $c^M$ as the interpretations of the symbols in the language.

We write the structure as $\mathcal{M} = (M, f^M, R^M, c^M : f \in F, R \in R, c \in C)$.

Warning about notational abuse: We use the convention that if $A$ is an $L$-structure, that $A$ will be its underlying set. However, often times when we are referring to an element $x \in A$, that is, an element of the universe, we may write $x \in A$. Effort should be made towards avoiding this, though it is an easy mistake to make in practice, and the meaning is generally clear for context.

Examples:

1. Let $\mathcal{L} = \{+, -, \cdot , <, 0, 1\}$. $\mathcal{M} = (\mathbb{R}, +, - , \cdot , <, 0, 1)$ where all of the symbols are interpreted in the usual way on $\mathbb{R}$.

2. Let $\mathcal{L} = \{E\}$. If we consider a, undirected simple graph with vertex set $G = \{v_1, \ldots , v_n\}$, and interpret $E$ by $E(v_i, v_j)$ if and only if there is an edge between $v_i$ and $v_j$, then $\mathcal{G} = (G, E)$ is an $L$-structure.

   Alternatively, consider $\mathbb{N}$ and say $E(n, m)$ if and only if $n < m$. Then $\mathcal{N} = (\mathbb{N}, E)$ is also an $L$-structure.

   The same language can be interpreted in many different ways. Note that these structures do not share any common features other than that they are described using a binary relation.

   We could also consider a structure with universe $\mathbb{N}$, but say that $E(n, m)$ if and only if $n|m$. Then $\mathcal{N}' = (\mathbb{N}, E)$ is an $L$-structure as well, with the same universe as $\mathcal{N}$, but different interpretation of the symbols in $\mathcal{L}$.

3. Let $\mathcal{L} = \emptyset$. Then for any non-empty set $M$, $\mathcal{M} = (M)$ is an $L$-structure.

4. Let $\mathcal{L} = \{+, c, 0 | c \in \mathbb{Q}\}$. Then consider the set of vectors $V = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z \in \mathbb{Q} \}$.

   This is a $\mathbb{Q}$-vector space, if we interpret $0$ as the 0-vector, and addition and scalar multiplication component-wise, then $\mathcal{V} = (V, +, c, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} _{c \in \mathbb{Q}}$ is an $L$-structure.

2.2 Terms and Formulas

Now that we have a way to describe specific classes of structures, we want to develop a way to articulate mathematical statements as we did with propositional logic.

Let $\mathcal{L}$ be a language with function symbols $\mathcal{F}$, relation symbols $\mathcal{R}$, and constant symbols $\mathcal{C}$.

The alphabet consists of the following symbols:
Now we will need two syntactic categories:

**Definition 18.** The set of *terms* in a language $L$, denoted $TERM$, is the smallest set $X$ satisfying the following:

1. $c \in X$ for all constant symbols $c \in C$, and $x_i \in X$ for all variables $x_i$.
2. If $f \in F$ and $t_1, \ldots, t_n \in X$, then $f(t_1, \ldots, t_n) \in X$.

The idea is that the terms will describe elements of the universe. Now we want to say things about these terms.

**Definition 19.** The set of first order formulas in a language $L$, denoted $FORM$, is the smallest set $X$ satisfying the following:

1. $\bot \in X$
2. If $t_1, \ldots, t_n \in TERM$ and $R \in \mathcal{R}$, $R(t_1, \ldots, t_n) \in X$
3. If $\phi, \psi \in X$, then $(\phi \Box \psi) \in X$ for $\Box \in \{\land, \lor, \rightarrow, \leftrightarrow\}$
4. If $\phi \in X$, then $(\neg \phi) \in X$.
5. If $\phi \in X$, then $(\forall x_i)\phi \in X$ and $(\exists x_i)\phi \in X$ for any variable $x_i$.

The formulas in item 1. are *atoms*.

As with $PROP$, and with very similar justification, we can prove things by induction on terms and formulas.

**Lemma 30.** Let $A(t)$ be a property of terms. If $A(c)$ is true for all $c \in C$, $A(x_i)$ is true for all variables $x_i$, and $A(t_1), \ldots, A(t_n) \Rightarrow A(f(t_1, \ldots, t_n))$ for all $n$-ary function symbols $f \in F$ and terms $t_1, \ldots, t_n$, then $A(t)$ holds for all $t \in TERM$.

**Proof.** Left as an exercise to the reader.

**Lemma 31.** Let $A(\phi)$ be a property of formulas. If

- $A(\phi)$ for $\phi$ atomic
• $A(\phi), A(\psi) \Rightarrow A(\phi \Box \psi)$

• $A(\phi) \Rightarrow A(\neg \phi)$

• $A(\phi) \Rightarrow A((\forall x_i)\phi), A((\exists x_i)\phi)$ for all variables $x_i$,

then $A(\phi)$ holds for all $\phi \in \text{FORM}$.

Proof. Left as an exercise to the reader.

Notational conventions/abbreviations:

• We will use the same guidelines for parentheses as in PROP. If we can avoid ambiguity, we may drop the parentheses. We also delete the brackets around $\forall x$ and $\exists x$ whenever possible.

• Quantifiers bind more strongly than the connectives.

• We may use $\forall x_1 x_2 \ldots x_n$ as shorthand for $\forall x_1 \forall x_2 \ldots \forall x_n$. Similarly for $\exists$.

• We will always assume that $n$ refers to the appropriate arity when we write $f(t_1, \ldots, t_n)$ or $R(t_1, \ldots, t_n)$.

• We will often overload $=$, since $=$ is a logical symbol but it also may appear in the meta language to define terms or formulas. When it is ambiguous, we will use $\hat{=} = \hat{\sim}$ to indicate that we are using $=$ as a logical symbol.

Examples: Consider a language with a binary relation $R$, binary function $f$, unary function $g$, and one constant symbol $c$. Examples of terms:

1. $t_1 = x_0$
2. $t_2 = f(x_1, x_2)$
3. $t_3 = g(c)$
4. $t_4 = f(g(x_1), x_4)$
5. $t_5 = f(f(g(c), g(c)), c)$

Examples of formulas:

1. $x_0 = x_2$
2. $t_3 = t_4$
3. $R(t_1, t_5)$
4. $(x_0 = x_1 \rightarrow x_1 = x_0)$
5. $(\forall x_0)(\exists x_1)f(x_0, x_1) = c$
6. $(\exists x_1)(\neg(x_1 = c) \land (g(x_1) = c))$

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We may also define functions on TERM and FORM by recursion, as with PROP. We will leave the details of this to the reader.

Next we want to examine which variables are associated to quantifiers, and which are not. We give the following recursive definition:

**Definition 20.** The set $FV(t)$ of free variables of a term $t$ is defined as

1. $FV(x_i) := \{x_i\}$
   $FV(c) := \emptyset$
2. $FV(f(t_1, \ldots, t_n)) = FV(t_1) \cup \ldots \cup FV(t_n)$.

**Definition 21.** The set $FV(\phi)$ of free variables of a formula $\phi$ is defined by

1. $FV(R(t_1, \ldots, t_n)) = FV(t_1) \cup \ldots \cup FV(t_n)$
   $FV(t_1 = t_2) = FV(t_1) \cup FV(t_2)$
   $FV(\bot) = \emptyset$
2. $FV(\phi \square \psi) = FV(\phi) \cup FV(\psi)$
   $FV(\neg \phi) = FV(\phi)$
3. $FV(\forall x_i \phi) = FV(\phi) \setminus \{x_i\}$
4. $FV(\exists x_i \phi) = FV(\phi) \setminus \{x_i\}$.

If a variable in $\phi$ appears with a quantifier, we call it a bound variable. We leave it as an exercise to the reader to give a definition of the set $BV(\phi)$ of bound variables.

**Definition 22.** A term $t$ or formulas $\phi$ is closed if $FV(t) = \emptyset$, or $FV(\phi) = \emptyset$ respectively. A closed formula is also called a sentence. A formula without quantifiers is called open. We use $TERM_C$ to denote the set of closed terms (terms with no variables) and $SENT$ to denote the set of sentences.

**Examples:**

1. $FV(t_1) = FV(x_0) = \{x_0\}$
2. $FV(t_2) = FV(f(x_1, x_2)) = FV(x_1) \cup FV(x_2) = \{x_1, x_2\}$
3. $FV(t_3) = FV(g(c)) = FV(c) = \emptyset$ (so this is a closed term)
4. $FV(t_4) = FV(f(g(x_1), x_4)) = FV(g(x_1)) \cup FV(x_4) = FV(x_1) \cup \{x_4\} = \{x_1, x_4\}$
5. $FV(t_5) = FV(f(f(g(c), g(c)), c))$
   $= FV(f(g(c), g(c))) \cup FV(c)$
   $= FV(g(c)) \cup FV(g(c)) \cup \emptyset$
   $= FV(c) \cup FV(c)$
   $= \emptyset \cup \emptyset = \emptyset$ (also a closed term)

Examples of formulas:
1. $\text{FV}(x_0 = x_2) = \text{FV}(x_0) \cup \text{FV}(x_2) = \{x_0, x_2\}$
2. $\text{FV}(t_3 = t_4) = \text{FV}(t_3) \cup \text{FV}(t_4) = \emptyset \cup \{x_1, x_4\} = \{x_1, x_4\}$
3. $\text{FV}(R(t_1, t_5)) = \text{FV}(t_1) \cup \text{FV}(t_5) = \{x_0\} \cup \emptyset = \{x_0\}$
4. $\text{FV}(x_0 = x_1 \to x_1 = x_0)$
   $= \text{FV}(x_0 = x_1) \cup \text{FV}(x_1 = x_0)$
   $= \text{FV}(x_0) \cup \text{FV}(x_1) \cup \text{FV}(x_1) \cup \text{FV}(x_0)$
   $= \{x_0, x_1\}$
5. $\text{FV}((\forall x_0)(\exists x_1)(f(x_0, x_1) = c))$
   $= \text{FV}((\exists x_1)(f(x_0, x_1) = c)) \setminus \{x_0\}$
   $= (\text{FV}(f(x_0, x_1) = c) \setminus \{x_1\}) \setminus \{x_0\}$
   $= ((\text{FV}(f(x_0, x_1)) \cup \text{FV}(c)) \setminus \{x_1\}) \setminus \{x_0\}$
   $= ((\text{FV}(x_0) \cup \text{FV}(x_1) \cup \emptyset) \setminus \{x_1\}) \setminus \{x_0\}$
   $= (\{x_0, x_1\} \setminus \{x_1\}) \setminus \{x_0\}$
   $= \{x_0\} \setminus \{x_0\} = \emptyset$. So this is a sentence.
6. $\text{FV}((\exists x_1)(\neg(x_1 = c) \land (g(x_1) = c)))$
   $= \text{FV}(\neg(x_1 = c) \land (g(x_1) = c) \setminus \{x_1\}$
   $= (\text{FV}(\neg(x_1 = c)) \cup \text{FV}(g(x_1) = c)) \setminus \{x_1\}$
   $= (\text{FV}(x_1 = c) \cup \text{FV}(g(x_1)) \cup \text{FV}(c)) \setminus \{x_1\}$
   $= (\text{FV}(x_1) \cup \text{FV}(c) \cup \text{FV}(x_1) \cup \emptyset) \setminus \{x_1\}$
   $= (\{x_1\} \cup \emptyset \cup \{x_1\}) \setminus \{x_1\}$
   $= \{x_1\} \setminus \{x_1\}$
   $= \emptyset$. So this is a sentence.

Warning: $\text{FV}(\phi)$ and $\text{BV}(\phi)$ may not be disjoint. For example, in the formula $(\forall x_1(x_1 = x_2)) \to P(x_1)$, $x_1$ is both free and bound, since the occurrence of $x_1$ in $P(x_1)$ is not within the scope of the quantifier.

We can define substitution analogously to our definition of substitution for PROP:

**Definition 23.** Let $s$ and $t$ be terms, then for a variable $x$, $s[t/x]$ is defined as follows

(i) If $s = y$, then

$$s[t/x] = y[t/x] = \begin{cases} y & y \neq x \\ t & t = x \end{cases}$$

That is, if $y$ and $x$ are the same variables, we replace $y$ with $t$, otherwise we do nothing.

If $s = c$ for some constant $c$, then $s[t/x] = c[t/x] = c$
(ii) If \( s = f(t_1, \ldots, t_p) \) for some \( n \)-ary function \( f \) and some terms \( t_1, \ldots, t_p \), then \( s[t/x] = f(t_1[t/x], \ldots, t_p[t/x]) \)

**Definition 24.** For a formula \( \phi \), \( \phi[t/x] \) is defined by:

(i) \( \bot [t/x] = \bot \)

\( R(t_1, \ldots, t_p)[t/x] = R(t_1[t/x], \ldots, t_p[t/x]) \) for relations \( R \) and terms \( t_1, \ldots, t_p \)

\( (t_1 \sim t_2)[t/x] = t_1[t/x] \sim t_2[t/x] \)

(ii) \( (\phi \land \psi)[t/x] = \phi[t/x] \land \psi[t/x] \)

\( (\neg \phi)[t/x] = \neg \phi[t/x] \)

(iii) \( (\forall y \phi)[t/x] = \forall y \phi[t/x] \) if \( x \neq y \), or \( \forall y \phi \) if \( x = y \) (since in this case, \( y \) is not free, so we do not replace it).

\( (\exists y \phi)[t/x] = \forall y \phi[t/x] \) if \( x \neq y \), or \( \exists y \phi \) if \( x = y \).

**Examples:**

1. \( t_4[t_2/x_1] = (f(g(x_1, x_4)))[f(x_1, x_2)/x_1] \)
   \( = f(g(x_1, x_4))f(x_1, x_2)/x_1) = f(g(x_1[f(x_1, x_2)/x_1], x_4[f(x_1, x_2)/x_1])) \)
   \( = f(g(f(x_1, x_2), x_4)) \).

2. \( (x_0 \sim x_2)[t_3/x_0] = (x_0 \sim x_2)[g(c)/x_0] = x_0[g(c)/x_0] \sim x_2[g(c)/x_0] = g(c) \sim x_2. \)

3. \( ((\forall x)f(x) = 0)[t/x] = (\forall x)f(x) = 0 \)

4. \( ((\exists y)R(y, x))[t/x] \)
   \( = (\exists y)R(y, x)[t/x] \)
   \( = (\exists y)R(y[t/x], x[t/x]) \)
   \( = (\exists y)R(y, t). \)

We can also consider simultaneous substitutions. We will leave writing the formal definition down as an exercise to the reader, but the idea is that \( \phi[t_1, \ldots, t_n/y_1, \ldots, y_n] \) results from simultaneously replacing each unbounded instance of \( y_i \) with \( t_i \).

Note that this is not the same thing as repeated substitution. For example,

\( (x_0 \sim x_1)[x_1 \sim x_0/x_0, x_1] \)
\( = (x_0[x_1 \sim x_0/x_0, x_1] \sim x_1[x_1 \sim x_0/x_0, x_1]) \)
\( = x_1 \sim x_0, \)

whereas \((x_0 \sim x_1)[x_1/x_0])[x_0/x_1] \)
\( = (x_0[x_1/x_0] \sim x_1[x_1/x_0])[x_0/x_1] \)
\( = (x_1 \sim x_1)[x_0/x_1] = x_1[x_0/x_1] \sim x_1[x_0/x_1] \)
\( = x_0 \sim x_0. \)

The quantifier clause in the definition prevents of from making substitutions for bounded variables, since this could change the meaning of a formula. For example, if we have \( \exists x(x = 0) \) and we substitute \( [y/x] \), \( \exists x(y = 0) \) is not the same statement. Furthermore, it is not enough to simply rename the variable next to the quantifier, since we may be substituting
a term other than a variable. For example, if we have a constant \(c\) and a unary function \(f\), 
\((\exists x (x = 0)) [f(c)/x]\) cannot be expressed as \(\exists f(c)(f(c) = 0)\).

There is one other case we need to avoid, which is a substitution in which a free variable becomes a bounded one after substitution. For example, \(\exists x (y < x)[x/y] = \exists x (y[x/y] < x[x/y]) = \exists x (x < x)\). While the first statement may be true in an ordered set, the second statement will not be. When performing substitutions, our goal is to preserve the truth value of the statement. We need to make this restriction precise as follows:

**Definition 25.** A term \(t\) is free for \(x\) in \(\phi\)

(i) \(\phi\) is atomic

(ii) \(\phi = \phi_1 \square \phi_2\) (or \(\phi = \neg \phi_1\)) and \(t\) is free for \(x\) in \(\phi_1\) and \(\phi_2\) (respectively, \(\phi_1\))

(iii) \(\phi = \exists y \psi\) (or \(\phi = \forall y \psi\)) and if \(x \in \text{FV}(\phi)\), then \(y \notin \text{FV}(t)\) and \(t\) is free for \(x\) in \(\psi\).

**Examples:**

1. \(x_2\) is free for \(x_0\) in \(\exists x_3 P(x_0, x_3)\)
2. \(f(x_0, x_1)\) is not free for \(x_0\) in \(\exists x_1 P(x_0, x_3)\)
3. \(x_5\) is free for \(x_1\) in \(P(x_1, x_3) \rightarrow \exists x_1 Q(x_1, x_2)\).

In summary, \(t\) is free for \(x\) in \(\phi\) as long as the free variables in \(t\) are not going to become bound variables after substitution in \(\phi\).

**Lemma 32.** If \(x \in \text{FV}(\phi)\), then \(t\) is free for \(x\) in \(\phi\) if and only if the free variables of \(t\) in \(\phi[t/x]\) are not bound by a quantifier.

**Proof.** \(\Rightarrow\): We will show by induction on \(\phi\) that if \(x \in \text{FV}(\phi)\) and \(t\) is free for \(x\) in \(\phi\), then \(\text{FV}(t) \cap \text{BV}(\phi[t/x]) = \emptyset\).

Base case: If \(\phi\) is atomic, then \(\phi[t/x]\) has no quantifiers, so \(\text{BV}(\phi[t/x]) = \emptyset\). Thus, \(\text{FV}(t) \cap \text{BV}(\phi[t/x]) = \emptyset\).

Induction Hypothesis: Assume the claim holds for some \(\phi\) and \(\psi\).

Assume \(x \in \text{FV}(\phi \square \psi)\) and \(t\) is free for \(x\) in \(\phi \square \psi\) where \(\square\) is a binary connective. Then \(t\) is free for \(x\) in both \(\phi\) and \(\psi\). So by IH, \(\text{FV}(t) \cap \text{BV}(\phi[t/x]) = \emptyset\) and \(\text{FV}(t) \cap \text{BV}(\psi[t/x]) = \emptyset\).

Then \(\text{FV}(t) \cap \text{BV}((x \square \psi)[t/x]) = \text{FV}(t) \cap \text{BV}(\phi[t/x] \square \psi[t/x]) = \text{FV}(t) \cap (\text{BV}(\phi[t/x]) \cup \text{BV}(\psi[t/x])) = (\text{FV}(t) \cap \text{BV}(\phi[t/x])) \cup (\text{FV}(t) \cap \text{BV}(\psi[t/x])) = \emptyset \cup \emptyset = \emptyset\).

Next, assume \(x \in \text{FV}(\neg \phi)\) and \(t\) is free for \(x\) in \(\neg \phi\). Then \(t\) is free for \(x\) in \(\phi\), so by IH, \(\text{FV}(t) \cap \text{BV}(\phi[t/x]) = \emptyset\). Then \(\text{FV}(t) \cap \text{BV}((\neg \phi)[t/x]) = \text{FV}(t) \cap \text{BV}(\neg \phi[t/x]) = \text{FV}(t) \cap \text{BV}(\phi[t/x]) = \emptyset\).

Now suppose \(x \in \text{FV}(Q y \phi)\) and \(t\) is free for \(x\) in \(Q y \phi\) where \(Q\) is a quantifier. Since \(x\) is free in \(Q y \phi\), \(x \neq y\). So \(\phi[t/x] = (Q y \phi)[t/x] = Q y \phi[t/x]\). Since \(t\) is free for \(x\) in \(Q y \phi\), \(y \notin \text{FV}(t)\) and \(t\) is free for \(x\) in \(\phi\). So by IH, \(\text{FV}(t) \cap \text{BV}(\phi[t/x]) = \emptyset\). Then \(\text{FV}(t) \cap \text{BV}((Q y \phi)[t/x]) = \text{FV}(t) \cap \text{BV}(Q y \phi[t/x]) = \text{FV}(t) \cap (\text{BV}(\phi[t/x]) \cup \{y\}) = (\text{FV}(t) \cap \text{BV}(\phi[t/x])) \cup (\text{FV}(t) \cap \{y\}) = \emptyset \cup \emptyset = \emptyset\).

\(\Leftarrow\): We will show by induction on \(\phi\) that if \(x \in \text{FV}(\phi)\) and if \(\text{FV}(t) \cap \text{BV}(\phi[t/x]) = \emptyset\), then \(t\) is free for \(x\) in \(\phi\).

Base case: If \(\phi\) is atomic, then \(t\) is free for \(x\) in \(\phi\).

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Induction Hypothesis: Assume the claim holds for some $\phi$ and $\psi$.

Assume $x \in FV(\phi \square \psi)$ and $FV(t) \cap BV((\phi \square \psi)[t/x]) = \emptyset$ where $\square$ is a binary connective. Then $FV(t) \cap BV((\phi \square \psi)[t/x]) = FV(t) \cap BV(\phi[t/x] \square \psi[t/x]) = FV(t) \cap BV(\phi[t/x] \square \psi[t/x]) = FV(t) \cap BV(\phi[t/x]) \cup BV(\psi[t/x]) = (FV(t) \cap BV(\phi[t/x])) \cup (FV(t) \cap BV(\psi[t/x])) = \emptyset$. Thus, we must have $FV(t) \cap BV(\phi[t/x]) = \emptyset$ and $FV(t) \cap BV(\psi[t/x]) = \emptyset$. So by IH, $t$ is free for $x$ in $\phi$ and $\psi$, so $t$ is free for $x$ in $\phi \square \psi$.

Now assume $x \in FV(\neg \phi)$ and $FV(t) \cap BV((\neg \phi)[t/x]) = \emptyset$. Then $FV(t) \cap BV((\neg \phi)[t/x]) = FV(t) \cap BV(\neg \phi[t/x]) = FV(t) \cap BV(\phi[t/x]) = \emptyset$, so by IH, $t$ is free for $x$ in $\phi$. Thus, $t$ is free for $x$ in $\neg \phi$.

Finally, suppose $x \in FV(Qy \phi)$ and $t$ is free for $x$ in $Qy \phi$ where $Q$ is a quantifier. Since $x$ is free in $Qy \phi$, $x \neq y$, so $(Qy \phi)[t/x] = Qy \phi[t/x]$. Assume $FV(t) \cap BV((Qy \phi)[t/x]) = \emptyset$. Then $FV(t) \cap BV((Qy \phi)[t/x]) = FV(t) \cap BV(Qy \phi[t/x]) = FV(t) \cap BV(\phi[t/x] \cup \{y\}) = (FV(t) \cap BV(\phi[t/x])) \cup (FV(t) \cap \{y\}) = \emptyset$. Thus, $FV(t) \cap BV(\phi[t/x]) = \emptyset$, so by IH, $t$ is free for $x$ in $\phi$, and $FV(t) \cap \{y\} = \emptyset$, so $y \notin FV(t)$. Thus, $t$ is free for $x$ in $Qy \phi$.

Note that this lemma is not true if $x \notin FV(\phi)$. For example, let $\phi(x)$ be $\forall x(x = x)$. So $x \notin FV(\phi)$. Let $t = x$. $t$ is free for $x$ in $\phi$, since $x \notin FV(\phi)$. However, $FV(t) = x$ and $BV(\phi) = BV(\forall x(x = x)) = BV(x = x) \cup \{x\} = \emptyset \cup \{x\} = \{x\}$. Thus, $FV(t) \cap BV(\phi) = \{x\} \neq \emptyset$.

Good news about notation: From now on, we are going to tacitly assume that all of our substitutions are “free for” the appropriate variables. We will begin to use notation like $\phi(x, y, z)$ to denote that the free variables of $\phi$ are $x, y, z$, and use the notation $\phi(t_1, t_2, t_3)$ to denote $\phi[t_1, t_2, t_3/x, y, z]$.

### 2.3 Interpretations for First Order Logic

So far we have seen that our language gives us a way to “talk about” various elements (via terms) from our structures. In several different contexts going forward, we will find it useful to provide names for all of the elements of a given structure.

**Definition 26.** Let $\mathcal{L}$ be a language and $\mathcal{M}$ be an $\mathcal{L}$-structure with universe $M$. The extended language $\mathcal{L}_\mathcal{M}$ (sometimes written $\mathcal{L}(\mathcal{M})$) is the language we get from adding a constant symbol for every element of $M$. We may denote the new set of constants $\{c_a : a \in M\}$.

**Example:** Let $\mathcal{L} = \{\oplus, e\}$ where $\oplus$ is a binary operator and $e$ is a constant (we can use this language to describe groups). Let $\mathcal{M} = (\mathbb{Z}, +, 0)$, so $\mathcal{M}$ is an $\mathcal{L}$-structure. Then $\mathcal{L}_\mathcal{M} = \{\oplus, e\} \cup \{\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots\}$. Note that $e$ and $c_0$ are both interpreted as 0 in this language, which is fine.

We will use this interpret the terms, and ultimately, formulas of a given language in a particular structure. Let $\mathcal{L}$ be a language and let $\mathcal{M}$ be an $\mathcal{L}$-structure.

**Definition 27.** An interpretation of the closed terms of $\mathcal{L}_\mathcal{M}$ in $\mathcal{M}$ is a mapping $(\cdot)\mathcal{M} : TERM_e \to M$ satisfying the following:

(i) For a constant symbol $c$ in $\mathcal{L}$, $c^\mathcal{M}$ is the interpretation of $c$ in $\mathcal{M}$.

For $a \in M$, $c^\mathcal{M}_a = a$.  

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(ii) If \( t_1, \ldots, t_n \) are closed terms and \( f \) is an \( n \)-ary function symbol, \( (f(t_1, \ldots, t_n))^M = f(t_1^M, \ldots, t_n^M) \).

Interpretations tell us what the closed terms describe in the context of our \( \mathcal{L} \)-structure \( M \). All of this information comes immediately from the information given by \( M \) being an \( \mathcal{L} \)-structure.

Now we want to interpret sentences to give a way to express if they are true or false in the context of our structure. We do this via valuations.

**Definition 28.** An interpretation of the sentences \( \phi \) of \( \mathcal{L}_M \) is a mapping \( [\cdot]_M : \text{SENT} \to \{0, 1\} \) satisfying:

1. \( [\bot]_M = 0 \)
2. \( [R(t_1, \ldots, t_n)]_M = 1 \) if \( (t_1^M, \ldots, t_n^M) \in R^M \) (that is, the \( n \)-tuple of interpretations of the terms are in the set given by the interpretation of \( R \) in the structure \( M \)), and it is equal to 0 otherwise.
3. \( [t_1 = t_2]_M = 1 \) if \( t_1^M = t_2^M \), = 0 otherwise.

Note that in order for an atomic valuation to be a sentence, all of the terms appearing must be closed terms. That is, there must not be any variables appearing in these formulas.

4. \( [\phi \land \psi]_M = \min([\phi]_M, [\psi]_M) \)
5. \( [\phi \lor \psi]_M = \max([\phi]_M, [\psi]_M) \)
6. \( [\phi \rightarrow \psi]_M = \max(1 - [\phi]_M, [\psi]_M) \)
7. \( [\phi \leftrightarrow \psi]_M = 1 - |[\phi]_M - [\psi]_M| \)
8. \( [-\phi]_M = 1 - [\phi]_M \)

(iii) \( [\forall x \phi]_M = \min\{[\phi[a/x]]_M | a \in M\} \)
9. \( [\exists x \phi]_M = \max\{[\phi[a/x]]_M | a \in M\} \)

Note that item (ii) coincides with our definition of valuation for propositions (though we may have seen slightly different equivalent characterizations).

From this we can see that the universal quantifier can be thought of like a big (infinite) conjunction, and the existential quantifier can be thought of like a big (infinite) disjunction. However, we still do not allow for arbitrary infinite conjunctions or disjunctions.

**Definition 29.** If \( \phi \) is a sentence in a language \( \mathcal{L} \) and \( M \) is an \( \mathcal{L} \)-structure, we write \( M \models \phi \) and say \( M \) satisfies \( \phi \) or \( M \) models \( \phi \) if \( [\phi]_M = 1 \).

This is a slight departure from our use of the symbol \( \models \) in propositional logic, as our valuation is now determined by the \( \mathcal{L} \)-structure \( M \), rather than just considering any valuation which is true on a set of propositions. It is however similar in spirit. We want to think of \( M \models \phi \) as saying that when we are working within \( M \), \( \phi \) is true.

If \( \Gamma \) is a collection of sentences, we write \( M \models \Gamma \) to mean \( M \models \phi \) for all \( \phi \in \Gamma \). We will write \( \Gamma \models \phi \) if for all \( \mathcal{L} \)-structures \( M \), if \( M \models \Gamma \), then \( M \models \phi \). When \( \Gamma = \emptyset \), we write \( \models \phi \) to mean that \( M \models \phi \) for all \( \mathcal{L} \)-structures \( M \).
If $\mathcal{M} \models \phi$, we will say that $\mathcal{M}$ models $\phi$. If $\Gamma$ is a set of sentences, we will call $\Gamma$ a theory, and if $\mathcal{M} \models \Gamma$, $\mathcal{M}$ is a model of the theory $\Gamma$.

Finally, if $\phi(z_1, \ldots, z_k)$ has free variables $\{z_1, \ldots, z_k\}$, we may write $\mathcal{M} \models \phi(z_1, \ldots, z_k)$ to mean $\mathcal{M} \models \forall z_1 \ldots z_k \phi(z_1, \ldots, z_k)$.

**Definition 30.** Let $\phi(z_1, \ldots, z_k)$ be an $\mathcal{L}$ formula with $\text{FV}(\phi) = \{z_1, \ldots, z_k\}$. $\phi$ is **satisfied** by $a_1, \ldots, a_k \in \mathcal{M}$ if $\mathcal{M} \models \phi(a_1, \ldots, a_k)$. In this case we say that $\phi$ is **satisfiable** in $\mathcal{M}$.

Note that $\phi(z_1, \ldots, z_k)$ is satisfiable in $\mathcal{M}$ if and only if $\mathcal{M} \models \exists z_1 \ldots z_k \phi(z_1, \ldots, z_k)$.

The meaning of our connectives coincides with their interpretation here as follows:

**Lemma 33.** Let $\phi$ and $\psi$ be sentences.

(i) $\mathcal{M} \models \phi \land \psi$ if and only if $\mathcal{M} \models \phi$ and $\mathcal{M} \models \psi$

(ii) $\mathcal{M} \models \phi \lor \psi$ if and only if $\mathcal{M} \models \phi$ or $\mathcal{M} \models \psi$

(iii) $\mathcal{M} \models \neg \phi$ if and only if $\mathcal{M} \nmodels \phi$

(iv) $\mathcal{M} \models \phi \rightarrow \psi$ if and only if $\mathcal{M} \models \phi \Rightarrow \mathcal{M} \models \psi$

(v) $\mathcal{M} \models \phi \leftrightarrow \psi$ if and only if $\mathcal{M} \models \phi \Leftrightarrow \mathcal{M} \models \psi$

(vi) $\mathcal{M} \models \forall x \phi(x)$ if and only if $\mathcal{M} \models \phi(c_a)$ for all $a \in M$

(vii) $\mathcal{M} \models \exists x \phi(x)$ if and only if $\mathcal{M} \models \phi(c_a)$ for some $a \in M$.

**Proof.** Follows from the definition, left as an exercise to the reader. \qed

**Examples:** Let $\mathcal{L} = \{\oplus, e\}$. Consider $\mathcal{M} = (\mathbb{Z}, +, 0)$.

To see that $\mathcal{M} \models \forall x (x + e = x)$, let $a \in \mathbb{Z}$ be given, and let $c_a$ be its corresponding constant in $\mathcal{L}_\mathcal{M}$.

Then $(c_a + e)^\mathcal{M} = a + 0 = a$, and $(c_a)^\mathcal{M} = a$, so $(c_a + e)^\mathcal{M} = (c_a)^\mathcal{M}$. Thus, $\models [c_a + e = c_a]_\mathcal{M} = 1$, so $\mathcal{M} \models c_a + e = c_a$. Thus, since $a \in M$ was arbitrary, $\mathcal{M} \models \forall x (x + e = x)$.

We see that this level of formalism is a little bit laborious, so we will take advantage of facts like Lemma 33 to help us.

For example, to see that $\mathcal{M} \models \neg \forall x \exists y (x = y \oplus y)$, we need to show that $\mathcal{M} \nmodels \forall x \exists y (x = y \oplus y)$. In other words, we are showing that it is not the case that for all $a \in M$ there is $b \in M$ such that $(c_a = c_b \oplus c_b)^\mathcal{M}$, or in other words, it is not the case that for all $a \in M$ there exists $b \in M$ such that $a = b + b$. Now we have reduced this syntactic statement to showing a fact about the additive group of the integers, and for this we can use our ordinary notion of proof. In this case, let $a = 1$. Then for all $b \in M$, $b + b = 2b \neq 1$ since 1 is not divisible by 2. Thus, it is not the case that for all $a \in M$ there is $b \in M$ such that $a = b + b$.

In other words, $\mathcal{M} \models \neg \forall x \exists y (x = y \oplus y)$.

In fact, we have the following extension of DeMorgan’s laws for quantifiers.

**Theorem 34.** (i) $\models \neg \forall x \phi \leftrightarrow \exists x \neg \phi$

(ii) $\models \neg \exists x \phi \leftrightarrow \forall x \neg \phi$

(iii) $\models \forall x \phi \leftrightarrow \neg \exists x \neg \phi$
(iv) \( \vdash \exists x \phi \leftrightarrow \neg \forall x \neg \phi \)

**Proof.** We will prove (i). The proof of (ii) is very similar, and (iii) and (iv) follow from (i) and (ii).

Let \( FV(\forall x \phi) = \{z_1, \ldots, z_k\} \). So we need to show that for any \( L \)-structure \( M \),

\[ M \vDash \forall z_1 \ldots z_k (\neg \forall x \phi(x, z_1, \ldots, z_k) \leftrightarrow \exists \neg \phi(x, z_1, \ldots, z_k)). \]

Let \( M \) be an \( L \)-structure and let \( a_1, \ldots, a_k \in M \) be given.

Using Lemma 33,

\[ M \vDash \forall x \phi(z, c_{a_1}, \ldots, c_{a_k}) \]

\[ \iff M \vDash \forall x \phi(z, c_{a_1}, \ldots, c_{a_k}) \]

\[ \iff \text{it is not the case that for all } b \in M, M \vDash \phi(c_b, c_{a_1}, \ldots, c_{a_k}) \]

\[ \iff \text{for some } b \in M, M \vDash \neg \phi(c_b, c_{a_1}, \ldots, c_{a_k}) \]

\[ \iff M \vDash \exists y \neg \phi(x, c_{a_1}, \ldots, c_{a_k}) \]

Thus, \( M \vDash \forall z_1 \ldots z_k (\neg \forall x \phi(x, z_1, \ldots, z_k) \leftrightarrow \exists \neg \phi(x, z_1, \ldots, z_k)). \)

Hence, \( M \vDash \forall z_1 \ldots z_k (\neg \forall x \phi(x, z_1, \ldots, z_k) \leftrightarrow \exists \neg \phi(x, z_1, \ldots, z_k)). \)

We also note that the order of quantification when we are not mixing quantifiers is irrelevant, and that quantifying over variables which are not free has no effect.

**Theorem 35.** (i) \( \vdash \forall x \forall y \phi \leftrightarrow \forall y \forall x \phi \)

(ii) \( \vdash \exists x \exists y \phi \leftrightarrow \exists y \exists x \phi \)

(iii) \( \vdash \forall x \phi \leftrightarrow \phi \text{ if } x \notin FV(\phi) \)

(iv) \( \vdash \exists x \phi \leftrightarrow \phi \text{ if } x \notin FV(\phi) \)

**Proof.** Left as an exercise to the reader.

As we have seen, \( \forall \) behaves like a generalization of \( \land \) and \( \exists \) behaves like a generalization of \( \lor \), so we get the following results:

**Theorem 36.** (i) \( \vdash \forall x (\phi \land \psi) \leftrightarrow (\forall x \phi \land \forall x \psi) \)

(ii) \( \vdash \exists x (\phi \lor \psi) \leftrightarrow (\exists x \phi \lor \exists x \psi) \)

(iii) \( \vdash \forall x (\phi(x) \lor \psi) \leftrightarrow \forall x \phi(x) \lor \psi \text{ if } x \notin FV(\psi) \)

(iv) \( \vdash \exists x (\phi(x) \land \psi) \leftrightarrow \exists x \phi(x) \land \psi \text{ if } x \notin FV(\psi) \).

**Proof.** (i) and (ii) follow immediately from the definition (details left as an exercise to the reader).

(iii) Let \( FV(\forall x (\phi(x) \lor \psi)) = \{z_1, \ldots, z_k\} \). We need to show that for any \( L \)-structure \( M \),

\[ M \vDash \forall z_1 \ldots z_k [\forall x (\phi(x) \lor \psi) \leftrightarrow \forall x \phi(x) \lor \psi]. \]

Let \( a_1, \ldots, a_n \in M \) be given. By Lemma 33, it will be enough to show that \( M \vDash \forall \phi [\phi(x, c_{a_1}, \ldots, c_{a_k}) \lor \psi(c_{a_1}, \ldots, c_{a_k})] \) if and only if \( M \vDash \forall x \phi(x, c_{a_1}, \ldots, c_{a_k}) \lor \psi(c_{a_1}, \ldots, c_{a_k}) \).

\[ \implies \text{ Let } b \in M \text{ be given. Then by Lemma 33, if } M \vDash \forall x (\phi(x, c_{a_1}, \ldots, c_{a_k}) \lor \psi(c_{a_1}, \ldots, c_{a_k})), \text{ then } M \vDash \phi(c_b, c_{a_1}, \ldots, c_{a_k}) \lor \psi(c_{a_1}, \ldots, c_{a_k}). \]

Then, we have at least one of \( M \vDash \phi(c_b, c_{a_1}, \ldots, c_{a_k}) \) or \( M \vDash \psi(c_{a_1}, \ldots, c_{a_k}) \). If the latter holds, then we get \( M \vDash \psi(c_{a_1}, \ldots, c_{a_k}) \).
∀xφ(x, c_{a_1}, ..., c_{a_k}) \lor ψ(c_{a_1}, ..., c_{a_k}), so we are done. Otherwise, \( M \not\models ψ(c_{a_1}, ..., c_{a_k}) \), which means that for every \( b \in M \), \( M \models φ(c_b, c_{a_1}, ..., c_{a_k}) \). Thus, \( M \models ∀xφ(x, c_{a_1}, ..., c_{a_k}) \), so \( M \models ∀xφ(x, c_{a_1}, ..., c_{a_k}) ← ψ(c_{a_1}, ..., c_{a_k}) \).

\[\Rightarrow:\] Using Lemma 33, \( M \models ∀xφ(x, c_{a_1}, ..., c_{a_k}) \lor ψ(c_{a_1}, ..., c_{a_k}) \)
\[\Rightarrow:\] \( M \models ∀xφ(x, c_{a_1}, ..., c_{a_k}) \lor ψ(c_{a_1}, ..., c_{a_k}) \)
\[\Rightarrow:\] \( M \models φ(c_b, c_{a_1}, ..., c_{a_k}) \lor ψ(c_{a_1}, ..., c_{a_k}) \), so \( M \models ∀x(φ(x, c_{a_1}, ..., c_{a_k}) \lor ψ(c_{a_1}, ..., c_{a_k})) \).

If \( M \models ψ(c_{a_1}, ..., c_{a_k}) \), then for any \( b \in M \), \( M \models φ(c_b, c_{a_1}, ..., c_{a_k}) \lor ψ(c_{a_1}, ..., c_{a_k}) \), so \( M \models ∀x(φ(x, c_{a_1}, ..., c_{a_k}) \lor ψ(c_{a_1}, ..., c_{a_k})) \).

Otherwise, if for all \( b \in M \), \( M \models φ(c_b, c_{a_1}, ..., c_{a_k}) \lor ψ(c_{a_1}, ..., c_{a_k}) \), then for all \( b \in M \), \( M \models φ(c_b, c_{a_1}, ..., c_{a_k}) \lor ψ(c_{a_1}, ..., c_{a_k}) \), so \( M \models ∀x(φ(x, c_{a_1}, ..., c_{a_k}) \lor ψ(c_{a_1}, ..., c_{a_k})) \).

(iv) This is similar to (iii) and left as an exercise.

\[\square\]

Warning: \( ∀x(φ(x) \lor ψ(x)) \) does not necessarily imply \( ∀xφ(x) \land ∀xψ(x) \), and \( ∃xφ(x) \land ∃xψ(x) \) does not necessarily imply \( ∃x(φ(x) \land ψ(x)) \).

Now that we have established some semantic means for our formulas via interpretations, we can work up to showing a variation of the substitution theorem.

**Lemma 37.** Let \( x, y \) be distinct variables and let \( r, s, t \) be terms.

(i) If \( x \notin \text{FV}(r) \), then \( (t[s/x])[r/y] = (t[r/y])[s/r/x] \)

(ii) If \( x \notin \text{FV}(s) \) and \( t \) and \( s \) are free for \( x \) and \( y \) in \( φ \), then \( (φ[t/x])[s/y] = (φ[s/y])[t[s/y]/x] \)

Proof. By induction on terms and induction of formulas respectively, left as an exercise to the reader.

\[\square\]

**Corollary 38.** Let \( M \) be an \( L \)-structure with universe \( M \). Let \( x, z \) be variables, \( t \) a term, an \( a \in M \).

(i) If \( z \notin \text{FV}(t) \), then \( t[c_a/x] = (t[z/x])[c_a/z] \)

(ii) If \( z \notin \text{FV}(φ) \) and \( z \) is free for \( x \) in \( φ \), then \( φ[c_a/x] = (φ[z/x])[c_a/z] \).

Example: Consider a language with a binary function \(+\) and a constant \( 0 \). Consider \( M = (\mathbb{Z}, 0, +) \) and the term \( t = x + x + y \). Let \( a = 1 \in \mathbb{Z} \). For the left hand side of the equation in part (i) of the Corollary, \( t[c_a/x] = (x + x + y)[c_1/x] = c_1 + c_1 + y \). On the right hand side, \( ((x + x + y)[z/x])[c_1/z] = (z + z + y)[c_1/z] = c_1 + c_1 + y \).

Using the corollary, we can show that it is possible to “pull out quantifiers” from a formula. This is a trick we use with bound variables in analysis. For example, \( \int xdx + \int sinydy = \int xdx + \int \sin xdx = \int x + \sin xdx \). This is an instance of “changing bound variables”.

**Theorem 39** (Change of Bound Variables). If \( x \) and \( y \) are free for \( z \) in \( φ \) and \( x, y \notin \text{FV}(φ) \), then \( \models ∃xφ[x/z] ⇐⇒ ∃yφ[y/z] \) and \( \models ∀xφ[x/z] ⇐⇒ ∀yφ[y/z] \).
Proof. We will prove the fact for the existential quantifier and leave the universal quantifier as an exercise to the reader (it is handled similarly).

It suffices to consider φ with $FV(φ) \subset \{z\}$. We need to show that for any structure $M$, $M \models \exists x φ[x/z] ⇔ M \models \exists φ[y/z]$. (As we have seen in previous proofs, if there are other free variables, replace the free variables with arbitrary constants corresponding to elements of $M$ in $L_M$.)

$$M \models \exists x φ[x/z]$$

$\iff M \models (φ[x/z])[a/x] \text{ for some } a \in M$

$\iff M \models φ[c_α/z] \text{ for some } a \in M \text{ (by the previous Corollary)}$

$\iff M \models (φ[y/z])[a/y] \text{ for some } a \in M \text{ (again, by the previous Corollary)}$

$\iff M \models \exists y φ[y/z]$, as required. \hfill \square$

The benefit of this theorem is that it allows us to replace a bound variable with a “fresh” one, to avoid the situation in which we have a variable which is both free and bound.

**Corollary 40.** Every formula is equivalent to one in which no variable occurs both free and bound.

**Theorem 41 (Substitution Theorem).** Let $t_1,t_2$ be terms.

(i) If $s$ is a term and $x$ is a variable, then $M \models t_1 = t_2 \rightarrow s[t_1/x] = s[t_2/x]$.

(ii) If $t_1$ and $t_2$ are free for $x$ in a formula $φ$, then $M \models t_1 = t_2 \rightarrow (φ[t_1/x] ⇔ φ[t_2/x])$.

**Proof.** Let $M$ be an $L$-structure and let $\{z_1,\ldots,z_k\}$ contain the free variables of $t_1$ and $t_2$. Let $a_1,\ldots,a_k \in M$ be given and let $t_1(\bar{c})$ denote $t_1[c_{a_1},\ldots,c_{a_k}/z_1,\ldots,z_k]$ and $t_2(\bar{c})$ denote $t_2[c_{a_1},\ldots,c_{a_k}/z_1,\ldots,z_k]$

For (i), we need to show that $M \models t_1(\bar{c}) = t_2(\bar{c}) \rightarrow s[t_1 (\bar{c})/x] = s[t_2 (\bar{c})/x]$. Assume $M \models t_1(\bar{c}) = t_2(\bar{c})$. If $s$ is a constant or a variable other than $x$, $(s[t_1 (\bar{c})/x])^M = s^M = (s[t_2 (\bar{c})/x])^M$. If $s = x$, then $(s[t_1 (\bar{c})/x])^M = (t_1(\bar{c}))^M = (t_2(\bar{c}))^M = (s[t_2 (\bar{c})/x])^M$. Either way, $M \models s[t_1 (\bar{c})/x] = s[t_2 (\bar{c})/x]$.

If $s$ is $f(s_1,\ldots,s_n)$ for some $n$-ary function symbol $f$ and terms $s_1,\ldots,s_n$ for which the claim holds, then $(s[t_1 (\bar{c})/x])^M = f^M((s_1[t_1 (\bar{c})/x])^M,\ldots,(s_n[t_1 (\bar{c})/x])^M)$ which is equal for $i = 1$ and $i = 2$ by IH. So $M \models (f(s_1,\ldots,s_n))[t_1 (\bar{c})/x] = (f(s_1,\ldots,s_n))[t_2 (\bar{c})/x]$.

For (ii), we need to show that for any formula $φ$, $M \models t_1(\bar{c}) = t_2(\bar{c}) \rightarrow (φ[t_1/x] ⇔ φ[t_2/x])$. Suppose $M \models t_1(\bar{c}) = t_2(\bar{c})$ and that $t_1$ and $t_2$ are free for $x$ in $φ$. It will suffice to assume that $FV(φ) = \{x\}$ (if not, as we have seen, we will substitute in arbitrary constants from $L_M$ corresponding to elements of $M$).

For $φ$ atomic, if $φ$ is $\bot$ then the claim holds vacuously. Consider $P(s_1,\ldots,s_n)$ where $P$ is a relation symbol. Then $M \models P(s_1,\ldots,s_n)[t_1(\bar{c})/x] ⇔ M \models P(s_1[t_1(\bar{c})/x],\ldots,s_n[t_1(\bar{c})/x])$. By part (i), $(s_1[t_1(\bar{c})/x])^M = (s_2[t_2(\bar{c})/x])^M$ for $1 \leq i \leq n$, so $(s_1[t_1(\bar{c})/x])^M,\ldots,(s_n[t_1(\bar{c})/x])^M \in P^M$ if and only if $((s_1[t_1(\bar{c})/x])^M,\ldots,(s_n[t_1(\bar{c})/x])^M) \in P^M$. Thus, this holds if and only if $M \models P(s_1[t_2(\bar{c})/x],\ldots,s_n[t_2(\bar{c})/x]) ⇔ M \models P(s_1,\ldots,s_n)[t_2(\bar{c})/x]$. So $M \models φ[t_1(\bar{c})/x] ⇔ φ[t_2(\bar{c})/x]$.

If $φ$ is $s_1 = s_2$, then $(s_1[t_1(\bar{c})/x])^M = (s_2[t_1(\bar{c})/x])^M$ and $(s_2[t_1(\bar{c})/x])^M = (s_2[t_2(\bar{c})/x])^M$.

By part (i), so $(s_1[t_1(\bar{c})/x])^M = (s_2[t_1(\bar{c})/x])^M$ if and only if $(s_1[t_2(\bar{c})/x])^M = (s_2[t_2(\bar{c})/x])^M$. Thus, $M \models s_1[t_1(\bar{c})/x] = s_2[t_1(\bar{c})/x] ⇔ s_1[t_2(\bar{c})/x] = s_2[t_2(\bar{c})/x]$.

Now suppose $φ_1$ and $φ_2$ are such that the claim holds.

$M \models (φ_1 \land φ_2)[t_1(\bar{c})/x]$
\[ M \models \phi_1[t_1(\bar{c})/x] \land \phi_2[t_1(\bar{c})/x] \]
\[ M \models \phi_1[t_1(\bar{c})/x] \land M \models \phi_2[t_1(\bar{c})/x] \text{ be Lemma 33,} \]
\[ M \models \phi_1[t_2(\bar{c})/x] \land M \models \phi_2[t_2(\bar{c})/x] \text{ by IH,} \]
\[ M \models \phi_1[t_2(\bar{c})/x] \land M \models \phi_2[t_2(\bar{c})/x] \]
\[ \forall, \rightarrow, \iff \text{ and } \neg \text{ are handled similarly, and left as an exercise to the reader.} \]

Now suppose \( \phi \equiv \exists y \psi \) where the claim holds for \( \psi \). Since \( t_1 \) and \( t_2 \) are free for \( x \) in \( \phi \), we know that \( x \in FV(\psi) \) and \( y \) is not a free variable in \( t_1 \) or \( t_2 \). In particular, \( t_1(\bar{c}) \) and \( t_2(\bar{c}) \) are both not equal to \( y \).

Let \( M \models (\exists y \psi)[t_1(\bar{c})/x] \)
\[ M \models (\exists y \psi)[t_1(\bar{c})/x] \]
\[ M \models (\exists y \psi)[t_2(\bar{c})/x] \]
\[ \iff M(\exists y \psi)[t_2(\bar{c})/x] \]

The universal quantifier is handled similarly and left as an exercise to the reader.

\[ \square \]

As in propositional logic, we will use the notation \( \phi \equiv \psi \) to denote \( \models \phi \iff \psi \), and refer to this as logical equivalence.

**Definition 31.** A formula \( \phi \) is in prenex normal form if \( \phi \) consists of a (possibly empty) string of quantifiers followed by a quantifier free formula. We call such a \( \phi \) a prenex formula.

**Examples:**
\[ \exists x \forall y \exists v ((x = z \lor y = z) \to v < y) \] is a prenex formula in a language with a binary relation \(<\).
\[ \forall x \forall y \exists z (P(x,y) \land Q(y,z) \to P(z,z)) \] is a prenex formula in a language with two binary relations \( P, Q \).
\[ \exists x (P(x) \land \forall y P(y)) \] is not a prenex formula.

It turns out every formula has a logically equivalent prenex formula.

**Theorem 42.** For each \( \phi \) there is a prenex formula \( \psi \) such that \( \phi \equiv \psi \).

**Proof.** By induction on formulas.

If \( \phi \) is atomic, then \( \phi \) is quantifier free, so the claim holds.

Let \( \phi \) and \( \psi \) be formulas with \( \phi' \) and \( \psi' \) in prenex form such that \( \phi \equiv \phi' \) and \( \psi \equiv \psi' \). By Theorem 39, we can ensure that all of the bound variables in \( \phi' \) and \( \psi' \) are all different and that the bound variables in one formula do not appear free in the other. That is, \( \phi' = (Q_1 y_1) \ldots (Q_n y_n) \phi_1 \) where \( Q_i \in \{ \forall, \exists \} \) and \( \phi_1 \) is quantifier free, and \( \psi' = (Q'_1 z_1) \ldots (Q'_m z_m) \psi_1 \) where \( Q'_i \in \{ \forall, \exists \} \), and \( \psi_1 \) is quantifier free, and \( y_1, \ldots, y_n \notin FV(\psi_1) \) and \( z_1, \ldots, z_m \notin FV(\phi_1) \).

Then by Theorem 35 and Theorem 36, \( \phi' \land \psi' \equiv (Q_1 y_1) \ldots (Q_n y_n) \phi_1 \land (Q'_1 z_1) \ldots (Q'_m z_m) \psi_1 \equiv (Q_1 y_1) \ldots (Q_n y_n) (Q'_1 z_1) \ldots (Q'_m z_m) (\phi_1 \land \psi_1) \) (the details of this are left to the reader, and shown in an example below).
The same argument shows us the result for $\phi \lor \psi$.

For $\neg \phi$, $\neg \phi' \equiv (Q_1 y_1) \ldots (Q_n y_n) \neg \phi_1$ by Theorem 34, which is a prenex formula.

For $\rightarrow$, use Lemma 33 to see that for any $\mathcal{M}$, $\mathcal{M} \models \phi \rightarrow \psi$ if and only if $\mathcal{M} \not\models \phi \Rightarrow \mathcal{M} \models \psi$, which holds if and only if $\mathcal{M} \not\models \phi$ or $\mathcal{M} \not\models \psi$, or equivalently, $\mathcal{M} \models \neg \phi \lor \psi$.

So since $\phi \rightarrow \psi \equiv \neg \phi \lor \psi$, we can apply the previous cases to get the result.

Similarly for $\phi \leftrightarrow \psi \equiv (\phi \rightarrow \psi) \land (\psi \rightarrow \phi)$.

Finally, for any $\mathcal{M}$, $\mathcal{M} \models \forall x \phi$ if and only if for any $a \in \mathcal{M}$, $\mathcal{M} \models \phi[a/x]$, which holds if and only if $\mathcal{M} \models ((Q_1 y_1) \ldots (Q_n y_n) \phi_1)[c_a/x]$ for any $a \in \mathcal{M}$, or equivalently, $\mathcal{M} \models (\forall x)(Q_1 y_1) \ldots (Q_n y_n) \phi_1$. Since $\mathcal{M}$ was arbitrary, $\forall x \phi \equiv \forall x \phi'$, which is in prenex normal form.

Similarly for $\exists$. $\square$

Example: Consider a language with unary relations $R$ and $S$.

We will show that $\forall x R(x) \land \exists x S(x)$ is logically equivalent to $\forall x \exists y (R(x) \land S(y))$, which is in prenex normal form.

Let $\mathcal{M}$ be a structure in the language.

$\mathcal{M} \models \forall x R(x) \land \exists x S(x)$

$\iff \mathcal{M} \models \forall x R(x)$ and $\mathcal{M} \models \exists x S(x)$ by Lemma 33

$\iff \mathcal{M} \models \forall x R(x)$ and $\mathcal{M} \models \exists y S(y)$ by Theorem 39

$\iff \mathcal{M} \models \forall x R(x)$ and $\mathcal{M} \models \forall x \exists y S(y)$ by Theorem 35(iii), since $x \notin \text{FV}(\exists y S(y))$

$\iff \mathcal{M} \models \forall x R(x) \land \forall x \exists y S(y)$ by Theorem 36(iv), since $y \notin \text{FV}(R(c_a))$

$\iff \mathcal{M} \models \forall x \exists y (R(x) \land S(y))$ by Lemma 35(i)

$\iff$ for any $a \in \mathcal{M}$, $\mathcal{M} \models R(c_a) \land \exists y S(y)$ by Lemma 33

$\iff$ for any $a \in \mathcal{M}$, $\mathcal{M} \models \exists y (R(c_a) \land S(y))$ by Theorem 36(iv) since $y \notin \text{FV}(R(c_a))$

$\iff \mathcal{M} \models \forall x \exists y (R(x) \land S(y))$.

2.4 Natural Deduction for First Order Logic

We will extend the system of natural deduction from propositional logic by handling the quantifiers, and equality. As in propositional logic, we use $\Gamma \vdash \phi$ to denote that there exists a derivation of $\phi$ with uncanceled hypotheses in $\Gamma$.

We will start with $\forall$.

$$
\begin{array}{c}
\phi(x) \\
\hline
\forall x \phi(x) \\
\forall x \phi(x)
\end{array}
\quad
\begin{array}{c}
\forall x \phi(x) \\
\hline
\phi(t)
\end{array}
\quad
\begin{array}{c}
\hline
\forall E
\end{array}
$$

In $\forall I$, we require that $x$ may not occur free in any uncanceled hypotheses in the derivation of $\phi(x)$.

In $\forall E$, we require $t$ to be free for $x$.

In $\forall I$, we require this restriction to avoid having false statements which are derivable. For example, the following derivation has an illegal first step, since the $\forall I$ takes place before the $x = 0$ assumption gets cancelled. This is a subtle thing to pay attention to: that the $\forall I$ is illegal because the $x = 0$ has not been cancelled at the point at which we are introducing $\forall$, it is cancelled much later with the $\rightarrow$ introduction.
\[
\frac{[x = 0]^1}{∀I} \\
\frac{∀x(x = 0)}{→ I_1} \\
\frac{x = 0 \rightarrow ∀x(x = 0)}{(∀I)} \\
\frac{∀x(x = 0 \rightarrow ∀x(x = 0))}{(∀E)} \\
0 = 0 \rightarrow ∀x(x = 0)
\]

This would give us \(\vdash 0 = 0 \rightarrow ∀x(x = 0)\), but \(\not\vdash 0 = 0 \rightarrow ∀x(x = 0)\); consider any structure with any non-zero element.

To see why we need the restriction on \((∀E)\), consider the following derivation. The first application of \((∀E)\) is illegal, since \(y\) is not free for \(x\) in \(¬∀y(x = y)\).

\[
\frac{∀x¬∀y(x = y)]^1}{(∀E)} \\
\frac{¬∀y(y = y)}{→ I_1} \\
\frac{∀x¬∀y(x = y) \rightarrow ¬∀y(y = y)}{}
\]

This will be false in any structure with at least two distinct elements, since given \(x\), it is not the case that for all \(y\) \(x = y\), but \(∀y(y = y)\), so the conclusion is false.

We can state these rules in terms of the derivability relation.

\((∀I)\) is \(Γ \vdash \phi(x) \Rightarrow Γ \vdash ∀xϕ(x)\) if \(x \notin FV(ψ)\) for all \(ψ ∈ Γ\)

\((∀E)\) is \(Γ \vdash ∀xϕ(x) \Rightarrow Γ \vdash ϕ(t)\) if \(t\) is free for \(x\) in \(ϕ\).

**Example:**

We can show \(\vdash ∀x(ϕ \land ψ) → ∀xϕ \land ∀xψ\) is derivable using natural deduction.

\[
\frac{∀x(ϕ(x) \land ψ(x))]^1}{(∀E)} \\
\frac{[∀x(ϕ(x) \land ψ(x))]^1}{(∀E)} \\
\frac{ϕ(x) \land ψ(x)}{(∧E)} \\
\frac{ϕ(x)}{(∀I)} \\
\frac{∀xϕ(x)}{(∀I)} \\
\frac{ϕ(x) \land ψ(x)}{(∀E)} \\
\frac{ψ(x)}{(∧E)} \\
\frac{∀xψ(x)}{(∀I)} \\
\frac{∀xϕ \land ∀xψ}{(∧I)} \\
\frac{∀xϕ \land ∀xψ}{(→ I_1)} \\
\frac{∀x(ϕ \land ψ) → ∀xϕ \land ∀xψ}{(→ I_1)}
\]

Note that our use of \((∀E)\) is legal because \(x\) is free for itself in \(ϕ(x)\), and in each instance of \((∀I)\), the only uncanceled hypothesis at that point is \(∀x(ϕ(x) \land ψ(x))\), in which \(x\) does not appear free.

Now we need to set things up so that we can prove the completeness theorem for first order logic by relating \(\vdash\) and \(\models\).

**Definition 32.** Let \(M\) be an \(L\)-structure with universe \(M\). Let \(Γ\) be a set of formulas and \(σ\) be a formula. Let \(\{x_1, x_2, \ldots\} = \bigcup_{ψ ∈ Γ} FV(ψ) ∪ FV(σ)\). Suppose \(a = (a_1, a_2, \ldots, a_n)\) is a sequence
of elements from \( M \) (repetitions allowed). Then \( \Gamma(a) \) is obtained from \( \Gamma \) by taking each \( \psi \) in \( \Gamma \) and simultaneous substituting \( x_i \) with \( c_{a_i} \), and \( \sigma(a) \) is the result of simultaneously substituting the relevant free variables in \( \sigma \) by the appropriate constants associated with \( a \).

(i) \( M \models \Gamma(a) \) if \( M \models \psi \) for all \( \psi \in \Gamma(a) \)

(ii) \( \Gamma \models \sigma \) if \( M \models \Gamma(a) \Rightarrow M \models \sigma(a) \) for all \( M \) and \( a \).

The idea is that \( \Gamma \models \sigma \) if for all structure \( M \) and all choices of \( a \), \( \sigma(a) \) is true in \( M \) as long as \( \Gamma(a) \) is true in \( M \).

If all of the formulas involved are sentences, we can just write \( \Gamma \models \sigma \) to mean \( M \models \Gamma(a) \Rightarrow M \models \sigma(a) \) for all \( M \) and \( a \).

We will need one small lemma before we proceed.

Lemma 43. If \( \Gamma' \subset \Gamma \) and \( \Gamma' \models \phi \), then \( \Gamma \models \phi \).

Proof. To show that \( \Gamma \models \phi \), let \( M \) and \( a \) a sequence of elements in \( M \) be given. Suppose \( M \models \Gamma(a) \). Then since \( \Gamma' \subset \Gamma \), \( M \models \Gamma'(a) \), so by assumption, \( M \models \phi(a) \). Thus, \( \Gamma \models \phi \).

Now we can state soundness.

Lemma 44 (Soundness). \( \Gamma \vdash \sigma \Rightarrow \Gamma \models \sigma \).

Proof. As in propositional logic, we proceed by induction on derivations. That is, if \( D \) is a derivation with uncanceled hypotheses in \( \Gamma \) and conclusion \( \sigma \), then \( \Gamma \vdash \sigma \).

Note that our definition of satisfaction is written in terms of interpretation, which contain propositional logic as a special case, since interpretations are in fact valuations. So the cases of a single element derivation (the base case) and the derivations with a propositional rule (\( \land \) introduction or elimination, \( \rightarrow \) introduction or elimination, \( \bot \), or RAA) we get from Soundness for propositional logic. It remains to check (\( \forall I \)) and (\( \forall E \)).

Let \( \phi(x) \) be a derivation for which the claim holds.

\[
\begin{array}{c}
\phi(x) \\
\hline
D
\end{array}
\]

\( \forall x \phi(x) \) is a derivationsuch that \( D \) has uncanceled hypotheses

in \( \Gamma \) and \( x \) is not free anywhere in the uncanceled hypotheses.

Let \( \Gamma' \subset \Gamma \) be the set of uncanceled hypotheses in \( D \). So \( x \) does not appear free in any formulas of \( \Gamma' \).

\( \Gamma' \vdash \phi(x) \), so by IH, \( \Gamma' \vdash \phi(x) \).

Let \( x_1 = x \) and \( \{x_2, x_3, \ldots\} = \bigcup_{\psi \in \Gamma'} FV(\psi) \cup FV(\forall x \phi(x)) \). Note that \( x_1 \) does not appear anywhere in the set \( \{x_2, x_3, \ldots\} \).

Let \( M \) be given and let \( a = (a_2, a_3, \ldots) \) be a sequence of elements from \( M \). We need to show that if \( M \models \Gamma'(a) \), then \( M \models (\forall x \phi(x))(a) \).

Assume \( M \models \Gamma'(a) \) and let \( a_1 \in M \) be given and let \( a' = (a_1, a_2, a_3, \ldots) \). \( \Gamma'(a') \) is the result of simultaneously substituting \( c_{a_i} \) for \( x_i \), so since \( x_1 \) does not appear free in any formulas of \( \Gamma' \), \( \Gamma'(a') = \Gamma'(a) \). Thus, \( M \models \Gamma'(a') \).
So since $\Gamma \vDash \phi(x)$, $M \vDash \phi(x)(a')$, that is, $M \vDash \phi(c_{a_1})(a)$. So, since $a_1 \in M$ was arbitrary, $M \vDash \forall x(\phi(x)(a))$, and thus, since $x$ does not appear amongst the variables which $a$ are replacing, $M \vDash \forall x \phi(x)(a)$ as required.

Now suppose we have a derivation $\forall x \phi(x)$ with uncanceled hypotheses in $\Gamma$ for which the claim holds. That is, $\Gamma \vDash \forall x \phi(x)$.

Let $\forall x \phi(x)$ be a term free for $x$ in $\phi$. Since $\Gamma \vDash \forall x \phi(x)$, then for any $M$ and $a$ in $M$, if $M \vDash \Gamma(a)$, then $M \vDash (\forall x \phi(x))(a)$.

Let $a$ be given and suppose $M \vDash \Gamma(a)$. So $M \vDash (\forall x \phi(x))(a)$. For ease of notation, let $(x, x_{i_1}, \ldots, x_{i_k})$ be the free variables of $\phi$ and let and let $\overline{a} = (c_{a_{i_1}}, \ldots, c_{a_{i_k}})$ be the corresponding elements of $a$.

So in other words, we know $M \vDash \forall x \phi(x, \overline{a})$. Now let $t$ be a term which is free for $x$ in $\phi$. Then if $(x_{i_1}, \ldots, x_{i_k})$ are the variables from $\{x_1, x_2, \ldots\}$ appearing in $t$, $t[c_{a_{i_1}}, \ldots, c_{a_{i_k}}/x_{i_1}, \ldots, x_{i_k}]$ is still free for $x$ in $\phi$ (since replacing variables with constants will not introduce any new bound variables).

Thus, we know that $M \vDash \phi(t[c_{a_{i_1}}, \ldots, c_{a_{i_k}}/x_{i_1}, \ldots, x_{i_k}], \overline{a})$, that is, $M \vDash (\phi(t))(a)$.

Hence, $\Gamma \vDash \phi(t)$.

\[\Box\]

### 2.5 Natural Deduction and Identity

Observe that we have restricted our attention to a framework in which we allow $=$ as a logical symbol, so we do not need to define an equality relation in any of our structures. We adopt the convention that the symbol $=$ is always interpreted as equality.

We see that the following axioms of equality hold in any structure:

1. (I$_1$) $\forall x (x = x)$
2. (I$_2$) $\forall x \forall y (x = y \to y = z)$
3. (I$_3$) $\forall x \forall y \forall z (x = y \land y = z \to x = z)$
4. (I$_4$) For any term $t$, $\forall x_1 \ldots x_n y_1 \ldots y_n (\bigwedge_{1 \leq i \leq n} x_i = y_i \to t(x_1, \ldots, x_n) = t(y_1, \ldots, y_n)$ and for any formula $\phi$, $\forall x_1 \ldots x_n y_1 \ldots y_n (\bigwedge_{1 \leq i \leq n} x_i = y_i \to (\phi(x_1, \ldots, x_n) \leftrightarrow \phi(y_1, \ldots, y_n)))$.

(I$_1$), (I$_2$), and (I$_3$) can be easily verified by checking that for any language $L$, they hold in any $L$-structure $M$.

For (I$_4$), we may need to take the universal closure. That is, if $FV(\forall x_1 \ldots x_n y_1 \ldots y_n (\bigwedge_{1 \leq i \leq n} x_i = y_i \to t(x_1, \ldots, x_n) = t(y_1, \ldots, y_n))) = \{z_1, \ldots, z_k\}$, we need to show $\forall z_1 \ldots z_k \forall x_1 \ldots x_n y_1 \ldots y_n (\bigwedge_{1 \leq i \leq n} x_i = y_i \to t(x_1, \ldots, x_n) = t(y_1, \ldots, y_n)$.
This then follows from the fact that in any language $L$ and any $L$-structure $M$, a term $t(\overline{x}, \overline{z})$ defines a function from $M^{k+n} \rightarrow M$. So for any $\overline{a}, \overline{b} \in M^n$, as long as $\overline{a} = \overline{b}$, $(t(c_{a_1}, \ldots, c_{a_n}, c_{d_1}, \ldots, c_{d_k}))^M = (t(c_{b_1}, \ldots, c_{b_n}, c_{d_1}, \ldots, c_{d_k}))^M$.

The latter part of (I4) is shown by induction on formulas, and is left as an exercise to the reader.

Note that (I4) is actually an infinite collection of axioms, we call this an *axiom schema*. We can express these axioms in terms of rules for natural deduction as follows:

(COMING SOON!)

### 2.6 Model Existence and the Completeness Theorem for First Order Logic

In order to prove the completeness theorem for first order logic, we will show that for any consistent set of sentences $\Gamma$, there exists a model $M \models \Gamma$. This will show that $\Gamma \models \phi \Rightarrow \Gamma \vdash \phi$, since if $\Gamma \not\models \phi$, then $\Gamma \cup \{\neg \phi\}$ is consistent, so there is $M \models \Gamma \cup \{\neg \phi\}$, but if $\Gamma \models \phi$, then since $M \models \Gamma$, $M \not\models \neg \phi$, which is impossible.

Throughout this section, we will assume that the formulas are sentences unless otherwise specified. We will also adopt the convention that if $M$ is a structure and $a \in M$, we will write $M \models \phi(a)$ to mean $M \models \phi(c_a)$.

We will not construct a model, much like in propositional logic, by way of considering maximally consistent sets of sentences.

**Definition 33.** (i) A theory $T$ is a collection of sentences which is closed under derivability. That is, if $T \vdash \phi$, then $\phi \in T$.

(ii) A set $\Gamma$ such that $T = \{\phi | \Gamma \vdash \phi\}$ is called an axiom set of the theory $T$. The elements of $\Gamma$ are axioms.

(iii) $T$ is called a *Henkin theory* if for each sentence $\exists x \phi(x)$, there is a constant $c$ such that $\exists x \phi(x) \rightarrow \phi(c) \in T$. This $c$ is called a witness for $\exists x \phi(x)$.

Observe that if $\Gamma$ is a set of sentences, $T = \{\sigma | \Gamma \vdash \sigma\}$ is a theory. Given $\phi$ such that $T \vdash \phi$, there are uncanceled hypotheses $\phi_1, \ldots, \phi_k \in T$. These all have derivations with uncanceled hypotheses in $\Gamma$, so putting these together we get a derivation of $\phi$ with uncanceled hypotheses in $\Gamma$, so $\phi \in T$.

**Definition 34.** Let $T$ and $T'$ be theories in languages $L$ and $L'$.

(i) $T'$ is an *extension* of $T$ if $T \subset T'$

(ii) $T'$ is a *conservative extension* of $T$ if $T' \cap L = T$. That is, all of the theorems in $T'$ which are in the language $L$ are already in $T$.

**Example:** Consider the language $L = \{+, 0\}$ and $L' = \{+, \cdot, -, 0, 1\}$ with three binary functions $+$, $\cdot$, and $-$, and two constants 0, 1. Let $T$ be $\forall x \forall y(x + y = y + x)$

\[ \forall x (x + 0 = x) \]
\[ \forall x \exists y (x + y = 0) \]
\[ \forall x \forall y \forall z (x + (y + z) = (x + y) + z) \]

And let $T'$ be $T$ along with
\(\forall x \forall y \forall z (x - y = z \leftrightarrow x = y + z)\)
\(\forall x (x \cdot 0 = 0)\)
\(\forall x \forall y \forall z (x \cdot (y \cdot z) = (x \cdot y) \cdot z)\)
\(\forall x (x \cdot 1 = x \wedge 1 \cdot x = x)\)
\(\forall x \forall y \forall z (x \cdot (y + z) = (x \cdot y) + (x \cdot z))\)
\(\forall x \forall y \forall z ((x + y) \cdot z = (x \cdot z) + (y \cdot z))\).

\(T'\) is a conservative extension of \(T\), since all of the sentences in \(T' \setminus T\) are not in the language \(\mathcal{L}\).

Looking ahead, we see that a Henkin theory will likely have a model, since the true existentially quantified sentences must have some witness in the structure (the interpretation of the constant). Our first goal is to extend consistent theories to Henkin theories.

**Definition 35.** Let \(T\) be a theory in a language \(\mathcal{L}\). Let \(\mathcal{L}^*\) be \(\mathcal{L}\) along with a fresh constant \(c_\phi\) for each sentence of the form \(\exists x \phi (x)\). \(T^*\) is the theory with axiom set \(T \cup \{ \exists x \phi (x) \rightarrow \phi (c_\phi) \mid \exists x \phi (x) \text{ is a sentence, with witness } c_\phi \}\).

**Lemma 45.** \(T^*\) is conservative over \(T\).

*Proof.* We will start by proving the following claim: Suppose \(\exists x \phi (x) \rightarrow \phi (c)\) is one of the new axioms. If \(\Gamma, \exists x \phi (x) \rightarrow \phi (c) \vdash \psi\) where \(\psi\) does not contain \(c\) and \(\Gamma\) is a set of sentences, none of which contain the constant \(c\), then \(\Gamma \vdash \psi\).

Once we have proved the claim, suppose \(T^* \vdash \psi\) where \(\psi\) is in the original language (that is, none of the constants \(c_\phi\) appear in \(\psi\)). Since derivations are finite, for some new axioms \(\sigma_1, \ldots, \sigma_n, T \cup \{ \sigma_1, \ldots, \sigma_n \} \vdash \psi\). Now we will show that \(T \vdash \psi\) by induction on \(n\). If \(n = 0\), we are done. Suppose \(T \cup \{ \sigma_1, \ldots, \sigma_n, \sigma_{n+1} \} \vdash \psi\) and let \(\Gamma' = T \cup \{ \sigma_1, \ldots, \sigma_n \}\). Then \(T', \sigma_{n+1} \vdash \psi\). Note that the constants appearing in each \(\sigma_i\) are distinct, so the new constant in \(\sigma_{n+1}\) does not appear anywhere in \(\Gamma'\) or \(\psi\). So, by the claim, \(\Gamma' \vdash \psi\). So by the induction hypothesis, \(T \vdash \psi\).

Here we outline the steps of the proof of the claim and leave the details as an exercise to the reader.

1. Since \(\Gamma, \{ \exists x \phi (x) \rightarrow \phi (c) \}\) \(\vdash \psi\), we get \(\Gamma \vdash (\exists x \phi (x) \rightarrow \phi (c)) \rightarrow \psi\) by \((\to I)\).

2. We can prove by induction on derivations that if \(y\) is a variables not appearing in \(\Gamma\) or \(\phi\), then \(\Gamma \vdash \phi \Rightarrow \Gamma[y/c] \vdash \phi[y/c]\).

3. So since \(c\) does not appear in \(\Gamma\) or \(\psi\), \(\Gamma[y/c] = \Gamma\) and \(\psi[y/c] = \psi\), so \(\Gamma \vdash (\exists x \phi (x) \rightarrow \phi (y)) \rightarrow \psi\), as long as we choose \(y\) such that it does not appear anywhere in the derivation.

4. \(\Gamma \vdash \forall y [(\exists x \phi (x) \rightarrow \phi (y)) \rightarrow \psi]\). This is an application of \((\forall I)\), which is valid since \(c\), and thus, \(y\), does not occur anywhere in \(\Gamma\) (so in particular, does not appear free in any uncanceled hypotheses).

5. \(\Gamma \vdash \exists y (\exists x \phi (x) \rightarrow \phi (y)) \rightarrow \psi\) (derivation coming soon).

6. \(\Gamma \vdash (\exists x \phi (x) \rightarrow \exists y \phi (y)) \rightarrow \psi\) (proof left as an exercise, use the fact that \(\exists x \phi (x) = \neg \forall x \neg \phi (x)\) and show \(\Gamma \vdash (\exists x \phi (x) \rightarrow \exists y \phi (y)) \equiv \exists y (\exists x \phi (x) \rightarrow \phi (y))\)).

7. \(\vdash \exists x \phi (x) \rightarrow \exists y \phi (y)\) (proof left as exercise).
8. \( \Gamma \vdash \psi \) by putting these together.

Note that our new \( T^* \) has a lot of witnesses, but we have also created a lot of new formulas by expanding our language, so \( T^* \) itself might not be a Henkin theory.

**Lemma 46.** Let \( T_0 := T \) and \( T_{n+1} := (T_n)^* \). Let \( T_\omega = \bigcup_{n<\omega} T_n \). Then \( T_\omega \) is a Henkin theory and is a conservative extension of \( T \).

**Proof.** Let \( \mathcal{L}_n \) denote the language of \( T_n \), and \( \mathcal{L}_\omega \) be their union (the language of \( T_\omega \)).

As before, we will outline the proof and leave the details as an exercise to the reader.

1. By induction on \( n \), we can show that \( T_n \) is a conservative extension of \( T \) for all \( n \).

2. To see that \( T_\omega \) is a theory, suppose \( T_\omega \vdash \sigma \), and let \( \{ \phi_0, \ldots, \phi_n \} \subset T_\omega \) be such that \( \{ \phi_0, \ldots, \phi_n \} \vdash \sigma \). Then each \( \phi_i \in T_{m_i} \) for some \( m_i \), so if we take \( k > m_i \) for all \( 0 \leq i \leq n \), \( \phi_i \in T_k \) since \( T_m \subset T_{m+1} \) for all \( m \). Then \( T_k \vdash \sigma \), so \( \sigma \in T_k \subset T_\omega \) since \( T_k \) is a theory.

3. To see that \( T_\omega \) is a Henkin theory, let \( \exists x \phi(x) \) be a formula in \( \mathcal{L}_\omega \). Then since formulas are finite, \( \exists x \phi(x) \) is in \( \mathcal{L}_n \) for some \( n \), so \( \exists x \phi(x) \rightarrow \phi(c) \in T_{n+1} \) by definition, since \( T_{n+1} = T_n^* \). Thus, \( \exists x \phi(x) \rightarrow \phi(c) \in T_\omega \).

4. \( T_\omega \) is conservative over \( T \), since if \( T_\omega \vdash \sigma \) if and only if \( T_n \vdash \sigma \) for some \( n \), so this follows from part 1.

**Corollary 47.** If \( T \) is consistent, then \( T_\omega \) is consistent.

**Proof.** If \( T_\omega \) is not consistent, then \( T_\omega \vdash \bot \). Since \( T_\omega \) is a conservative extension and \( \bot \) is in the language of \( T \), we must have \( T \vdash \bot \).

Now we want to extend \( T_\omega \) as far as possible, just like we did with maximally consistent theories in propositional logic.

**Lemma 48 (Lindenbaum).** Each consistent theory is contained in a maximally consistent theory.

**Proof.** Let \( T \) be a consistent theory. Consider the set \( A \) of all consistent extensions \( T' \) of \( T \). This set is partially ordered by inclusion. We will show that every chain has an upper bound. Let \( \{ T_i : i \in I \} \) be a chain of extensions of \( T \). Let \( T' = \bigcup_{i \in I} T_i \). We leave it as an exercise to the reader to show that \( T' \) is a consistent extension of \( T \).

\( T' \) is an upper bound of this chain, so by Zorn’s lemma, \( A \) has a maximal element, we will call it \( T_m \).

\( T_m \) is maximally consistent, since given any consistent extension \( T' \) of \( T \), if \( T_m \subset T' \), we must have \( T_m = T' \), since \( T_m \) is maximal with respect to inclusion.

Observe that maximally consistent extensions are by no means unique.
Lemma 49. An extension of a Henkin theory with the same language is again a Henkin theory.

Proof. If we do not expand the language, then in any extension $T'$ it is still the case that for any $\exists x \phi(x)$ there is a constant $c$ such that $T' \vdash \exists x \phi(x) \rightarrow \phi(c)$. \qed

Corollary 50. Every consistent theory has a maximally consistent extension which is a Henkin theory.

Proof. If $T$ is consistent, $T_\omega$ is a consistent Henkin theory, so it is contained in some maximally consistent theory which is also a Henkin theory. \qed

Lemma 51 (Model Existence Lemma). If $\Gamma$ is consistent, then $\Gamma$ has a model.

Proof. Let $T = \{\sigma | \Gamma \vdash \sigma\}$ be a theory with axiom set $\Gamma$. If $\mathcal{M} \models T$, then $\mathcal{M} \models \Gamma$.

Let $T_m$ be a maximally consistent Henkin extension of $T$. Let $\mathcal{L}_m$ be the language of $T_m$ (so the language may be expanded with constants from when we extended $T$ to a Henkin theory).

We will construct a model of $T_m$, which will be a model of $T$.

Let $A = \{t \in \mathcal{L}_m | t$ is a closed term\} (no free variables). For each function symbol $f$, define a function $\hat{f} : A^k \rightarrow A$ by $\hat{f}(t_1, \ldots, t_k) := f(t_1, \ldots, t_k)$. For each $n$-ary relation symbol $R$, define a relation $\hat{R} \subset A^n$ by $(t_1, \ldots, t_n) \in \hat{R}$ if and only if $T_m \vdash R(t_1, \ldots, t_n)$. For each constant symbol $c$, define a constant $\hat{c} = c$.

So far, $A = (A, \hat{f}, \hat{R}, \hat{c} : f, R, c \in \mathcal{L}_m)$ is an $\mathcal{L}_m$-structure, but we need to ensure that equality behaves correctly, which it may not in $A$. For example, we may have terms $s$ and $t$ which are equal according to $T_m$, but if they are not literally the same string of symbols, we would not view them as equal in this context. e.g. if $T_m \vdash 1 + 0 = 1$, then $1 + 0$ and $1$ may both be terms in $A$, but we would not view them as equal. That is, $A \models 1 + 0 = 1$.

In order to address this, we define a relation $t \sim s$ by $T_m \vdash t = s$ for $t, s \in A$, then $\sim$ is an equivalence relation. We know that $T_m \vdash I_1, I_2, I_3$, so using these we get that this relation satisfies reflexivity, symmetry, and transitivity.

If $t_i \sim s_i$ for $1 \leq i \leq n$, and $(t_1, \ldots, t_n) \in \hat{R}$, then $(s_1, \ldots, s_n) \in \hat{R}$, and $\hat{f}(t_1, \ldots, t_n) \sim \hat{f}(s_1, \ldots, s_n)$. These follow from $T_m \vdash I_4$.

Let $[t]$ denote the equivalence class of $t$ under $\sim$. Define $\mathcal{M} = (A/\sim, \hat{R}, \hat{f}, \hat{c} : R, f, c \in \mathcal{L}_m)$.

Define $\hat{R} := \{([t_1], \ldots, [t_n]) | (t_1, \ldots, t_n) \in \hat{R} \}$, $\hat{f}([t_1], \ldots, [t_n]) = [\hat{f}(t_1, \ldots, t_n)]$, and $\hat{c} = [\hat{c}]$.

Since $(t_1, \ldots, t_n) \in \hat{R} \Leftrightarrow (s_1, \ldots, s_n) \in \hat{R}$ and $f(t_1, \ldots, t_n) \sim f(s_1, \ldots, s_n)$ when $t_i \sim s_i$ for $1 \leq i \leq n$, these functions and relations are well defined.

By induction on (closed) terms, we see that $t^\mathcal{M} = [t]$. That is, these equivalence classes exactly coincide with the interpretations of the closed terms in this structure.

If $t = c$, then $t^\mathcal{M} = \hat{c} = [\hat{c}] = [t]$. (We do not need to consider $t = x$ a variable, since we only have closed terms.)

If $t = f(t_1, \ldots, t_k)$, then $t^\mathcal{M} = \hat{f}(t_1^\mathcal{M}, \ldots, t_k^\mathcal{M}) = \hat{f}([t_1], \ldots, [t_k])$ by IH, $= [\hat{f}(t_1, \ldots, t_k)] = [f(t_1, \ldots, t_k)]$.

Thus, for $[t]$ an element of the universe $A/\sim$, and its corresponding constant $c_{[t]}$ in the extended language, $\mathcal{M} \models \phi(t)$ if and only if $\mathcal{M} \models \phi(c_{[t]})$.

We leave it as an exercise to show by induction on formulas that $\mathcal{M} \models \phi(t)$ if and only if $\mathcal{M} \models \phi(c_{[t]})$, using the fact that $t^\mathcal{M} = c_{[t]}$.  

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Now we are ready to show that $\mathcal{M} \models \phi(t)$ if and only if $T_m \vdash \phi(t)$ in the language $\mathcal{L}_m$ of $T_m$. This is by induction on formulas.

For $\phi$ atomic: $\mathcal{M} \models R(t_1, \ldots, t_n)$ $\iff$ $(t_1^\mathcal{M}, \ldots, t_n^\mathcal{M}) \in \tilde{R}$ $\iff$ $([t_1], \ldots, [t_n]) \in \tilde{R}$ $\iff$ $(t_1, \ldots, t_n) \in \tilde{R}$ $\iff$ $T_m \vdash R(t_1, \ldots, t_n)$. Equality of closed terms follows from the same argument, and $\phi = \bot$ is trivial.

We leave $\phi \land \psi$, $\phi \rightarrow \psi$, and $\neg \phi$ as an exercise to the reader. They just use the induction hypothesis and properties of $\vdash$ and $\models$ with respect to the connectives.

Finally, suppose $\phi$ is $\forall x \psi(x)$. Then $\mathcal{M} \models \forall x \psi(x)$ $\iff$ for all $a \in A/ \sim$, $\mathcal{M} \models \psi(a)$.

So let $c$ be the witness belonging to $\exists x \neg \psi(x)$ (in the Henkin theory). Then $\mathcal{M} \models \psi(c)$. By IH, $T_m \vdash \psi(c)$. We also know that $T_m \vdash \exists x \neg \psi(x) \rightarrow \neg \psi(c)$, so we know $T_m \vdash \psi(c) \rightarrow \neg \exists x \neg \psi(x)$. That is, $T_m \vdash \forall x \phi(x)$.

Conversely, if $T_m \vdash \forall x \psi(x)$, then $T_m \vdash \psi(t)$ for all closed terms $t$ (since these will be free for $x$ in $\psi$ because they have no variables), so by IH, $\mathcal{M} \models \psi(t)$ for all closed terms $t$, that is, since these describe all elements of the universe $A/ \sim$, $\mathcal{M} \models \forall x \psi(x)$.

\[ \square \]

**Theorem 52** (Completeness). $\Gamma \models \phi \Rightarrow \Gamma \vdash \phi$.

**Proof.** If $\Gamma \not\models \phi$, then $\Gamma \cup \{ \neg \phi \}$ is consistent, so by the Model Existence Lemma, there is $\mathcal{M} \models \Gamma \cup \{ \neg \phi \}$. That is, $\mathcal{M} \models \mathcal{M}$ and $\mathcal{M} \models \neg \phi$, so $\mathcal{M} \not\models \phi$. Thus, $\Gamma \not\models \phi$. \[ \square \]

### 3 Model Theory

#### 3.1 Mappings Between Structures

Throughout this section, let $\mathcal{L}$ be a language.

**Definition 36.** Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures with universes $M$ and $N$ respectively. An $\mathcal{L}$-embedding $\eta: \mathcal{M} \rightarrow \mathcal{N}$ is a one-to-one map from $M \rightarrow N$ which preserves the interpretations of all of the symbols in $\mathcal{L}$ as follows:

Let $a_1, \ldots, a_n \in M$ be given.

- (i) $\eta(f^\mathcal{M}(a_1, \ldots, a_n)) = f^\mathcal{N}(\eta(a_1), \ldots, \eta(a_n))$ for $f$ a function symbol in $\mathcal{L}$,
- (ii) $(a_1, \ldots, a_n) \in R^\mathcal{M}$ if and only if $(\eta(a_1), \ldots, \eta(a_n)) \in R^\mathcal{N}$ for all relation symbols $R$ in $\mathcal{L}$,
- (iii) $\eta(c^\mathcal{M}) = c^\mathcal{N}$ for constant symbols $c$ in $\mathcal{L}$.

A bijective $\mathcal{L}$-embedding is called an $\mathcal{L}$-isomorphism. If $M \subset N$, the inclusion map $M \rightarrow N$ is an $\mathcal{L}$-embedding, and we say that $\mathcal{M}$ is a substructure of $\mathcal{N}$, or $\mathcal{N}$ is an extension of $\mathcal{M}$.

**Examples:**

In most algebraic structures, the notion of $\mathcal{L}$-embedding will coincide with the notion of homomorphism.

1. $(\mathbb{Z}, +, 0)$ is a substructure of $(\mathbb{R}, +, 0)$. 

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2. Let \( \mathcal{L} = \{ \oplus, 0, q \cdot | q \in \mathbb{Q} \} \) be a language with a binary function \( \oplus \), a constant 0, and unary function \( q \cdot \) for all \( q \in \mathbb{Q} \). We can view \( \mathbb{Q} \)-vector spaces as \( \mathcal{L} \)-structures by interpreting 0 as the 0-vector, \( \oplus \) as vector addition, and \( q \cdot \) as scalar multiplication.

Consider \( \mathcal{M} = (\mathbb{Q}^3, +, (0, 0, 0), q : q \in \mathbb{Q}) \) and \( \mathcal{N} = (\mathbb{Q}^4, +, (0, 0, 0, 0), q : q \in \mathbb{Q}) \). Let \( \eta((a, b, c)) = (a, b, c, 0) \). We can check that this is an \( \mathcal{L} \)-embedding:

\[ \eta \] is one-to-one, since if \( \eta((a, b, c)) = \eta((x, y, z)) \), then \( (a, b, c, 0) = (x, y, z, 0) \), so \( a = x, b = y, \) and \( c = z \). That is, \( (a, b, c) = (x, y, z) \).

\[ \eta(0^\mathcal{M}) = \eta((0, 0, 0)) = (0, 0, 0, 0) = 0^\mathcal{N} \]

\[ \eta((a, b, c) \oplus^\mathcal{M} (x, y, z)) = \eta((a + x, b + y, c + z)) = (a + x, b + y, c + z, 0) = (a, b, c, 0) \oplus^\mathcal{N} (x, y, z, 0) = \eta((a, b, c)) \oplus^\mathcal{N} \eta((x, y, z)) \]

\[ \eta(q \cdot^\mathcal{M} (a, b, c)) = \eta((qa, qb, qc)) = (qa, qb, qc, 0) = q \cdot^\mathcal{N} (a, b, c, 0). \]

Note that this coincides with a linear transformation.

3. In the same language, consider \( \mathcal{M} = (\mathbb{Q}^3, +, 0, q : q \in \mathbb{Q}) \). Let \( \mathcal{A} = (\{ ax^2 + bx + c | a, b, c \in \mathbb{Q} \}, +, 0, q : q \in \mathbb{Q}) \). \( \mathcal{A} \) is the \( \mathbb{Q} \)-vector space of polynomials of degree at most 2.

Let \( \eta((a, b, c)) = ax^2 + bx + c \). First note that \( \eta \) is both injective and surjective: If \( \eta((a, b, c)) = \eta((d, e, f)) \), then \( ax^2 + bx + c = dx^2 + ex + f \), so we must have \( a = d, b = e, \) and \( c = f \), that is \( (a, b, c) = (d, e, f) \). Given \( ax^2 + bx + c, (a, b, c) \mapsto ax^2 + bx + c \), so it is surjective.

Furthermore, this is an embedding:

\[ \eta(0^\mathcal{M}) = \eta((0, 0, 0)) = 0x^2 + 0x + 0 = 0 = 0^\mathcal{A} \]

\[ \eta((a, b, c) \oplus^\mathcal{M} (d, e, f)) = \eta((a + d, b + e, c + f)) = (a + d)x^2 + (b + e)x + (c + f) = (ax^2 + bx + c) \oplus^\mathcal{A} (dx^2 + ex + f) = \eta((a, b, c)) \oplus^\mathcal{A} \eta((d, e, f)). \]

\[ \eta(q \cdot^\mathcal{M} (a, b, c)) = \eta((qa, qb, qc)) = qa(x^2 + bx + c) = q^\mathcal{A} (ax^2 + bx + c) = q^\mathcal{A} \eta((a, b, c)). \]

Note that this coincides with \( \mathbb{Q} \)-vector space isomorphism.

4. Let \( \mathcal{L} = \{ \ast, e \} \) where \( \ast \) is a binary function and \( e \) is a constant. Let \( \mathcal{M} = (\mathbb{Z}, +, 0) \) and \( \mathcal{N} = (\mathbb{R}, \cdot, 1) \).

Let \( \eta : \mathcal{M} \to \mathcal{N} \) be given by \( \eta(x) = e^x \). To check that \( \eta \) is an \( \mathcal{L} \)-embedding, first note that if \( e^x = e^y \), then \( \ln(e^x) = \ln(e^y) \), so \( x = y \). That is, \( \eta \) is one-to-one.

\[ \eta(x \cdot^\mathcal{M} y) = \eta(x + y) = e^{x+y} = e^x \cdot e^y = \eta(x) \cdot^\mathcal{N} \eta(y) \]

\[ \eta(e^M) = \eta(0) = e^0 = 1 = e^N. \]

**Proposition 53.** Suppose \( \mathcal{M} \) is a substructure of \( \mathcal{N} \), \( \bar{a} \in \mathcal{M} \) and \( \phi(\bar{a}) \) is a quantifier-free formula. Then \( \mathcal{M} \models \phi(\bar{a}) \) if and only if \( \mathcal{N} \models \phi(\bar{a}) \).

**Proof.** First we will prove that if \( t(\bar{a}) \) is a term and \( \bar{b} \in \mathcal{M} \), then \( t^\mathcal{M}(\bar{b}) = t^\mathcal{N}(\bar{b}) \).

If \( t \) is constant, then \( t^\mathcal{M} = c^\mathcal{M} = c^\mathcal{N} = t^\mathcal{N} \).

If \( t \) is a variable \( v_i \), then \( t^\mathcal{M}(\bar{b}) = v_i = t^\mathcal{N}(\bar{b}) \), if \( t \) is any other variable \( x, t^\mathcal{M}(\bar{b}) = x = t^\mathcal{N}(\bar{b}) \).

If \( t = f(t_1, \ldots, t_n) \) and the claim holds for \( t_1, \ldots, t_n, \) then \( t^\mathcal{M} = f^\mathcal{M}(t_1^\mathcal{M}(\bar{b}), \ldots, t_n^\mathcal{M}(\bar{b})) = f^\mathcal{N}(t_1^\mathcal{M}(\bar{b}), \ldots, t_n^\mathcal{M}(\bar{b})) = f^\mathcal{N}(t_1^\mathcal{N}(\bar{b}), \ldots, t_n^\mathcal{N}(\bar{b})) \) by IH, \( t^\mathcal{N}(\bar{b}) \).

Now we will prove the proposition by induction on formulas.
If $\phi$ is $t_1 = t_2$, then
\[ M \models (t_1 M(\bar{a}) = t_2 M(\bar{a})) \]
\[ \iff (t_1 N(\bar{a}) = t_2 N(\bar{a})) \]
\[ \iff (by \ the \ first \ part) \ t_1 N(\bar{a}) = t_2 N(\bar{a}) \]
\[ \iff N \models (t_1(\bar{a}) = t_2(\bar{a})). \]

If $\phi$ is $R(t_1, \ldots, t_n)$, then
\[ M \models \phi(\bar{a}) \]
\[ \iff (t_1 M(\bar{a}), \ldots, t_n M(\bar{a})) \in R^M \]
\[ \iff (t_1 N(\bar{a}), \ldots, t_n N(\bar{a})) \in R^N \]
\[ \iff (by \ the \ first \ part) \ N \models R(t_1(\bar{a}), \ldots, t_n(\bar{a})) \]
\[ \iff N \models \phi(\bar{a}). \]

Now suppose the claim is true for some $\phi$ and $\psi$
\[ M \models \phi(\bar{a}) \land \psi(\bar{a}) \iff M \models \phi(\bar{a}) \land M \models \psi(\bar{a}) \iff N \models \phi(\bar{a}) \land N \models \psi(\bar{a}) \]
\[ \iff N \models \phi(\bar{a}) \land \psi(\bar{a}). \]

This is enough since $\phi \lor \psi \equiv \lnot (\lnot \phi \land \lnot \psi)$ and $\phi \to \psi \equiv \lnot \phi \lor \psi$. So we have the result for all quantifier free formulas.

\[ \square \]

Note that when $\phi$ is not quantifier free, this is not necessarily true. For example,
\[ (\mathbb{N}, \leq) \subset (\mathbb{Z}, \leq), \text{so consider } \phi(x) = \forall y(x \leq y) \text{ and } 0 \in \mathbb{N}. \ (\mathbb{N}, \leq) \models \forall y(0 \leq y), \text{however} \ (\mathbb{Z}, \leq) \not\models \forall y(0 \leq y). \]

We will be interested in the setting in which two structures satisfy all of the same sentences.

**Definition 37.** Let $M$ and $N$ be $\mathcal{L}$-structures. We say that $M$ and $N$ are **elementarily equivalent** and write $M \equiv N$ if for every sentence $\phi$ in $\mathcal{L}$, $M \models \phi$ if and only if $N \models \phi$.

Let $Th(M) = \{ \phi : \phi \text{ is an } \mathcal{L}\text{-sentence and } M \models \phi \}$, we call this the **theory of $M$**. Note $M \models Th(M)$, and that $Th(M)$ is a complete consistent theory: for every sentence $\phi$, either $M \models \phi$, or $M \not\models \phi$, so $M \models \lnot \phi$, and if $Th(M)$ was not consistent, we would have $Th(M) \models \bot$, so by soundness, $Th(M) \models \bot$, which would mean $M \models \bot$, which is impossible.

It is easy to see that $M \equiv N$ if and only if $Th(M) = Th(N)$.

**Theorem 54.** If $j : M \to N$ is an isomorphism, then $M \equiv N$. More specifically, for any $\mathcal{L}$ formula $\phi(x_1, \ldots, x_n)$ and $\bar{a} = (a_1, \ldots, a_n) \in M^n$, $M \models \phi(\bar{a}) \iff N \models \phi(j(\bar{a}))$.

**Proof.** We will show this by induction on formulas.

First, by induction on terms, we will show that for any term $t(v_1, \ldots, v_n)$ (where $v_1, \ldots, v_n$ are the free variables of $t$), for $\bar{a} = (a_1, \ldots, a_n) \in M^n$, $j(t M(\bar{a})) = t N(j(\bar{a}))$ where $j(\bar{a}) = (j(a_1), \ldots, j(a_n)) \in N^n$.

If $t = c$ a constant, then $t(\bar{a}) = t = c$, and $j(t M(\bar{a})) = j(c M) = c N = t N(j(\bar{a}))$ by the definition of $\mathcal{L}$-embedding.

If $t = v_i$, then $j(t M(\bar{a})) = j(a_i) = t N(j(\bar{a}))$. If $t = x$ where $x \neq v_i$ for $1 \leq i \leq n$, then $j(t M(\bar{a})) = j(x M) = x N = t N(j(\bar{a}))$.

Finally, if $t_1, \ldots, t_k$ are terms for which the claim is true, all with free variables among $v_1, \ldots, v_n$, then if $t(v_1, \ldots, v_n) = f(t_1(v_1, \ldots, v_n), \ldots, t_n(v_1, \ldots, v_n))$,
\[ j(t^M(\bar{a})) = j(f^M(t_1^M(\bar{a}), \ldots, t_n^M(\bar{a}))) \]
\[ = f^N(j(t_1^M(\bar{a})), \ldots, j(t_k^M(\bar{a}))) \text{ by the definition of } L\text{-embedding}, \]
\[ = f^N(t_1^N(j(\bar{a})), \ldots, t_n^N(j(\bar{a}))) \text{ by IH, } = t^N(j(\bar{a})). \]

Now we will prove the theorem.

For atomic formulas:
\[ M \vDash t_1(\bar{a}) = t_2(\bar{a}) \]
\[ \iff t_1^M(\bar{a}) = t_2^M(\bar{a}) \]
\[ \iff t_1^N(j(\bar{a})) = t_2^N(j(\bar{a})) \text{ by the first part,} \]
\[ \iff N \vDash t_1(j(\bar{a})) = t_2(j(\bar{a})). \]
\[ M \vDash R(t_1(\bar{a}), \ldots, t_k(\bar{a})) \]
\[ \iff (t_1^M(\bar{a}), \ldots, t_k^M(\bar{a})) \in R^M \]
\[ \iff (j(t_1^M(\bar{a})), \ldots, j(t_k^M(\bar{a}))) \in R^N \text{ since } j \text{ is an } L\text{-embedding,} \]
\[ \iff (t_1^N(j(\bar{a})), \ldots, t_k^N(j(\bar{a}))) \in R^N \text{ by the first part,} \]
\[ \iff N \vDash R(t_1(j(\bar{a})), \ldots, t_n(j(\bar{a}))). \]

Now let \( \phi(\bar{x}) \) and \( \psi(\bar{x}) \) be formulas for which the claim holds (note that we may need to expand the list of free variables \( \bar{x} \) to include the free variables of both \( \phi \) and \( \psi \)).
\[ M \vDash \neg \phi(\bar{a}) \]
\[ \iff M \not\vDash \phi(\bar{a}) \]
\[ \iff N \not\vDash \phi(j(\bar{a})) \text{ by IH,} \]
\[ \iff N \vDash \neg \phi(\bar{a}). \]
\[ M \vDash \phi(\bar{a}) \land \psi(\bar{a}) \]
\[ \iff M \vDash \phi(\bar{a}) \text{ and } M \vDash \psi(\bar{a}) \]
\[ \iff N \vDash \phi(j(\bar{a})) \text{ and } N \vDash \psi(j(\bar{a})) \text{ by IH,} \]
\[ \iff N \vDash \phi(j(\bar{a})) \land \psi(j(\bar{a})). \]
\[ M \vDash \forall x \phi(x, \bar{a}) \]
\[ \iff M \vDash \phi(b, \bar{a}) \text{ for all } b \in M, \]
\[ \iff N \vDash \phi(j(b), j(\bar{a})) \text{ for all } b \in M, \]
\[ \iff N \vDash \phi(c, j(\bar{a})) \text{ for all } c \in N, \text{ since } j \text{ is bijective, so all } c \in N \text{ appear as } j(b) \text{ for some } b \in M, \]
\[ \iff N \vDash \forall x \phi(x, j(\bar{a})). \]

\[ \square \]

**Definition 38.** If \( \sigma \in M \to M \) is an isomorphism, we call \( \sigma \) an *automorphism of \( M \).*

For example, consider \( M = (\mathbb{N}, \leq) \) and let \( \sigma : M \to M \) be given by \( \sigma(n) = n + 1. \) \( \sigma \) is an \( L\)-embedding, since for \( a, b \in \mathbb{N}, a \leq^M b \iff a + 1 \leq^M b + 1 \iff \sigma(a) \leq^M \sigma(b). \) Furthermore, \( \sigma \) is bijective, so \( \sigma \) is an isomorphism.

However, if we expand our language to include any constant \( c, \) we would require \( \sigma(c^M) = c^M, \) so this would no longer be an \( L\)-embedding.

### 3.2 Examples of Theories

For now, look at homework 7. Those examples will be put here soon!
3.3 Definable Sets

One major object of study for model theorists are the definable sets of a theory, that is, the sets which can be described using the language in models of a particular theory. As we have seen, first order logic is reasonable robust, in that we are able to express most of the mathematical concepts which we are interested in studying. However, we will see that in some cases, there are reasonably natural sets which we can describe in the meta-setting which are not definable. We need to study these carefully so that going forward, we understand which sets our model theoretic tools can be applied to.

Definition 39. Let $M$ be an $L$-structure with universe $M$. $X \subseteq M^n$ is definable if there is an $L$-formula $\phi(v_1, \ldots, v_n, w_1, \ldots, w_m)$ and $\bar{b} \in M^m$ such that $X = \{\bar{a} \in M^n : M \models \phi(\bar{a}, \bar{b})\}$. We say that $\phi(\bar{v}, \bar{b})$ defines $X$, and sometimes write $X = \phi(M, \bar{b})$.

We say that $X$ is $A$-definable or definable over $A$ for $A \subseteq M$ if there is a formula $\psi(\bar{v}, w_1, \ldots, w_l$ and $\bar{b} \in A^l$ such that $\psi(\bar{v}, \bar{b})$ defines $X$.

Examples:

1. Some from homework 7 (coming soon!)

2. Let $L = \{+, -, \cdot, 0, 1\}$ be the language of rings ($+,-,\cdot$ are all binary functions, 0, 1 are constants). Let $M = (\mathbb{R}, +, -, \cdot, 0, 1)$ be any $L$-structure (for example, $R = \mathbb{R}$, $+,-,\cdot, 0, 1$ interpreted in the usual way). Let $p(x) \in R[x]$, that is $p(x)$ is a polynomial of one variable with coefficients in $R$. So $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ for some $n \in \mathbb{N}$, $a_0, \ldots, a_n \in R$.

   The set $Y = \{x \in R : p(x) = 0\}$ is an example of a definable set. Let $\phi(x, v_0, \ldots, v_n)$ be the formula $0 = v_0 + v_1 \cdot x + \ldots + v_n \cdot x^n$ where $x^n$ is shorthand for $x \cdot \ldots \cdot x$ $n$-times. Let $\bar{a} = (a_0, \ldots, a_n)$.

   Then $x \in Y$ if and only if $M \models \phi(x, \bar{a})$.

   For any $A$ with $a_0, \ldots, a_n \in A$, $Y$ is $A$-definable.

3. Since we can view $n$-ary relations as subsets of $M^n$, we say that a relation is definable if the subset $M^n$ which determines the relation is a definable set.

   For example, consider $M = (\mathbb{R}, +, -, \cdot, 0, 1)$. Note that we do not have a symbol in the language for $<$. Let $\phi(x, y)$ be the formula $\exists z(z \neq 0 \land x + z^2 = y)$. We know that in $\mathbb{R}$, if $z \neq 0$, then $z^2 > 0$, so $x < y$ if and only if $y - x = z^2$ for some $z \neq 0$. In other words, $x < y$ if and only if $M \models \phi(x, y)$. The relation $<$ is definable.

4. Similarly, we may view an $n$-ary function as a set by looking at its graph. Consider a structure $M$ with universe $M$ and a function $f : M^k \to M$. $f$ is a definable function if $\{ (a_1, \ldots, a_k, b) \in M^{k+1} : f(a_1, \ldots, a_k) = b \}$ is a definable set.

   For example, as in our example where we examined solution sets of polynomials, we said to let $x^n$ be shorthand for $x \cdot \ldots \cdot x$ ($n$-times). $x^n$ is a definable function.

5. Let $M = (\mathbb{Q}, +, -, \cdot, 0, 1)$ be the field of rational numbers. Let $\phi(x, y, z)$ be the formula $\exists a \exists b \exists c(xy^2 + 2 = a^2 + xy^2 - yc^2)$.

   Let $\psi(x)$ be the formula $\forall y \forall z([\phi(y, z, 0) \land (\forall w(\phi(y, z, w) \rightarrow \phi(y, z, w+1)))] \rightarrow \phi(y, z, x))$. 

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A remarkable result of Julia Robinson (a model theorist) shows that $\mathcal{M} \models \phi(x)$ if and only if $x$ is an integer. (Think about why there is no obvious way to define the integers in this language.)

**Proposition 55.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure with universe $M$. If $X \subset M^n$ is $A$ definable, then every $\mathcal{L}$-automorphism of $\mathcal{M}$ that fixes $A$ pointwise fixes $X$ setwise. That is, if $\sigma(a) = a$ for all $a \in A$, then $X = \{\sigma(x) : x \in X\}$.

**Proof.** Let $\psi(\bar{v}, \bar{a})$ with $\bar{a} = (a_1, \ldots, a_n) \in A^n$ be the $\mathcal{L}$-formula defining $X$. Let $\sigma$ be an automorphism of $\mathcal{M}$ which fixes $A$ pointwise, so $\sigma(a_i) = a_i$ for $1 \leq i \leq n$. Let $\bar{b} \in M^n$.

By Theorem 54, $\mathcal{M} \models \psi(\bar{b}, \bar{a}) \iff \mathcal{M} \models \psi(\sigma(\bar{b}), \sigma(\bar{a})) \iff \mathcal{M} \models \psi(\sigma(\bar{b}), \bar{a})$.

Thus, $\bar{b} \in X$ if and only if $\sigma(\bar{b}) \in X$. \qed

This proposition gives us a lot of information about the possible automorphisms of a structure. For example, consider $\mathcal{M} = (\mathbb{R}, <)$ the closed interval from $-1$ to $1$ with the usual ordering and a constant interpreted as $0$. We can show that the set of rational number in $\mathcal{M}$ is not definable as follows. Suppose $\mathbb{Q}$ is definable by some $\phi(x, a_1, \ldots, a_n)$ where $a_1, \ldots, a_n$ are parameters in $\mathbb{R}$. Let $k \in \mathbb{Q}$ be such that $k > \max(a_1, \ldots, a_n)$. Consider the following function:

$$\sigma(r) = \begin{cases} r & r \leq k \\ k + \pi(q - k) & r > k \end{cases}$$

$\sigma$ is a bijection, and if $x < y$ then $\sigma(x) < \sigma(y)$, so $\sigma$ is in fact an automorphism of $\mathcal{M}$, and in particular, $\sigma$ fixes $\{a_1, \ldots, a_n\}$ pointwise. However, let $q \in \mathbb{Q}$ with $q > k$ be given. Then $\sigma(q) = k + \pi(q - k)$, which is not rational (if it was equal to some rational $a$, then we would get $\pi = \frac{a - k}{q - k} \in \mathbb{Q}$). Thus, $\sigma$ does not fix $\mathbb{Q}$ set wise, so $\mathbb{Q}$ cannot be definable.

Along with sets, we may want to consider the definability of a particular element of the universe.

**Definition 40.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subset M$. We say that $b$ is definable over $A$ if there is $\phi(x, y_1, \ldots, y_k)$ an $\mathcal{L}$-formula and $\bar{a} \in A^k$ such that $\mathcal{M} \models \phi(b, \bar{a}) \land \forall y(\phi(y, \bar{a}) \rightarrow y = b)$.

Let $dcl(A) = \{x \in M : x$ is definable over $A\}$. We call this the definable closure of $A$.

The idea is that we want to distinguish elements which may not be named by a constant but which we can describe using the language.

**Examples:**

1. The interpretation of any constant $c$ in a language is definable over $\emptyset$ via the formula $x = c$.
2. Let $\mathcal{M} = ([0, 1], <)$. $0$ is definable over $\emptyset$ via $\phi(x) = \forall y(x < y)$. Similarly, $1$ is definable by $\psi(x) = \forall y(y < x)$.
3. If $a \in A$, then $a \in dcl(A)$ (via the formula $x = a$).
4. $dcl(dcl(A)) = dcl(A)$ (left as an exercise to the reader).
5. If $a \in dcl(A)$ and $\sigma$ is an automorphism of $\mathcal{M}$ fixing $A$ point wise, then $\sigma(a) = a$. 

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We also have the more general notion of algebraic elements, generalizing the notion of being algebraic over a field.

**Definition 41.** We say that \(b \in M\) is algebraic over \(A\) if there exists an \(L\)-formula \(\phi(v, w_1, \ldots, w_k)\) and \(\bar{a} \in A^k\) such that \(M \models \phi(b, \bar{a})\) and \(\{y \in M : M \models \phi(y, \bar{a})\}\) is finite.

We let \(acl(A) = \{x : x\text{ is algebraic over } A\}\), and we call this the algebraic closure of \(A\).

**Examples:**

1. In \(L = \{+, \cdot, -, 0, 1\}\), if \(p(a) = 0\) where \(p(x) = a_0 + a_1 x + \ldots + a_n x^n\) is a polynomial over \(\mathbb{R}\), then \(a \in acl(\{a_0, \ldots, a_n\})\), since polynomials have finitely many roots.

2. In \((\mathbb{Z}, 0, \leq)\), given \(n < m\) in \(\mathbb{Z}\), if \(n \leq x \leq m\), then \(x \in acl(\{n, m\})\), since \(\phi(x) = n \leq x \land x \leq m\) has only finitely many elements which satisfy it.

3. \(dcl(A) \subset acl(A)\), since if \(x \in dcl(A)\), there is a formula with parameters from \(A\) such that the only element satisfying it is \(x\) (so this set is finite).

4. \(acl(acl(A)) = acl(A)\) (left as an exercise to the reader).

5. If \(A \subset B\), then \(acl(A) \subset acl(B)\) (left as an exercise to the reader).

### 3.4 The Compactness Theorem

Here we will prove one of the most important theorems in model theory: that if a theory \(T\) is finitely satisfiable, then \(T\) itself is satisfiable. Recall that an \(L\)-theory \(T\) is satisfiable if there is an \(L\)-structure \(M\) such that \(M \models T\). \(T\) is finitely satisfiable if for any finite \(\Delta \subset T\), there exists an \(L\)-structure \(M\) such that \(M \models \Delta\).

**Theorem 56 (Compactness Theorem).** *If \(T\) is finitely satisfiable, then \(T\) is satisfiable.*

**Proof.** Suppose \(T\) is not satisfiable. Then by Lemma 51, since \(T\) has no model, \(T\) is inconsistent, that is, \(T \vdash \bot\). Since derivations are finite, for some finite \(\Delta \subset T\), \(\Delta \vdash \bot\). So by Soundness, \(\Delta \models \bot\). So if \(\Delta\) was satisfiable, we would have \(M \models \bot\), which is impossible. Hence, \(T\) is not finitely satisfiable.

The compactness theorem can also be proved directly using Henkin constructions, similar to our proof of the completeness theorem. We will outline this for the reader later on.

The Compactness Theorem has many wide reaching applications, including non-standard models of arithmetic.

**Examples:**

1. Let \(L = \{+, \cdot, <, 0, 1\}\) be the language of arithmetic and let \(N = (\mathbb{N}, +, \cdot, <, 0, 1)\) be \(\mathbb{N}\) with the usual interpretations of the symbols. Let \(T = Th(\mathbb{N})\). We will use compactness to show that it is possible to find a model of \(Th(\mathbb{N})\) in which there is an “infinite element”.

   Let \(L^* = L \cup \{c\}\) where \(c\) is a new constant.

   Let \(n\) be shorthand for \(1 + 1 + \ldots + 1\) (\(n\)-times).

   Let \(\Gamma = T \cup \{n < c : n = 0, 1, 2, 3, \ldots\}\).
We will show that \( \Gamma \) is satisfiable by showing that it is finitely satisfiable. Let \( \Delta \subset \Gamma \) be finite. Then for some \( n \in \mathbb{N} \), \( \Delta \subset T \cup \{0 < c, 1 < c, \ldots, n < c\} \). Consider \( \mathcal{M} = (\mathbb{N}, +, \cdot, <, 0, 1, n + 1) \), where the new constant \( c \) is interpreted as \( n + 1 \). \( \mathcal{M} \models \Delta \). Thus, since \( \Gamma \) is finitely satisfiable, by compactness, it has some model \( \mathcal{M} \). Note that \( \mathcal{M} \) as an \( \mathcal{L} \)-structure is still a model of \( T \).

Let \( \omega \in M \) be the interpretation of this new constant \( c \). We can study the properties of \( \omega \) by examining the universally quantified statements which are true in \( T \). For example, \( T \models \forall x(x + 1 \neq x \land x + 1 > x) \), so we must have an element \( \omega + 1 \) in \( M \) such that \( \omega < \omega + 1 \).

More surprisingly, consider the induction axiom schema: for a formula \( \phi(x) \), \( (\phi(0) \land \forall x(\phi(x) \to \phi(x + 1))) \to \forall x\phi(x) \).

This is still true in \( \mathcal{M} \), which means that if we have a formula \( \phi \) such that \( \phi(0) \) and for all \( x \), \( \phi(x) \to \phi(x + 1) \), we will get \( \phi(\omega) \). This seems surprising, since our usual notion of induction does not support that we can “jump” to \( \omega \).

However, if \( \phi(0) \) and \( \forall x(\phi(x) \to \phi(x + 1)) \), then \( \mathcal{N} \models \forall x\phi(x) \), which means \( T \models \forall x\phi(x) \), and thus, since \( \mathcal{M} \models T \), \( \mathcal{M} \models \forall x\phi(x) \). Thus, we get \( \mathcal{M} \models \phi(\omega) \).

2. Let \( \mathcal{L} = (\circ, e) \) be the language of groups (\( \circ \) is a binary function, \( e \) is a constant).

Recall that in a group, the order of an element \( x \) is the number of times \( x \) must be multiplied by itself to get back to the identity. For example, let

\[ G = \{(1)(2)(3), (123), (132), (1)(2)(3), (1)(23), (13)(2)\} \]

be the set of permutations of 3 elements where \( e = (1)(2)(3) \) and \( \circ \) is interpreted as composition. Then the order of \( (123) \) is 3, since \( (123)(123)(123) \) will return the 3 elements to their original order. The order of \( (12)(3) \) is 2.

Alternatively, consider \( G = \mathbb{Z}/7\mathbb{Z} = \{0, 1, 2, 3, 4, 5, 6\} \) where \( \circ \) is interpreted as addition mod 7. That is, \( x \circ y = x + y (\text{mod} 7) \), and \( e = 0 \). Then the order of any non-zero element in \( G \) is 7, for example, \( 3 \circ 3 \circ 3 \circ 3 \circ 3 = 21 \mod 7 = 0 \).

Let \( T \) be the theory of groups, so \( T \) is axiomatized by \( \{\forall x(x \circ e = x \land e \circ x = x), \forall x\exists y(x \circ y = e \land y \circ x = e), \forall x\forall y\forall z(x \circ (y \circ z) = (x \circ y) \circ z)\} \).

We will use compactness to show that there is no formula \( \phi(x) \) such that \( \phi(x) \) holds if and only if \( x \) has finite order (we call such an \( x \) a torsion element).

Suppose there is such an \( \mathcal{L} \)-formula \( \phi(x) \). Expand the language with a new constant \( c \) and let \( \Gamma \) be the theory axiomatized by \( T \cup \{\phi(c)\} \cup \{c \circ \ldots \circ c \neq e : n = 1, 2, 3, \ldots\} \).

Note that if we have a model of \( \Gamma \), we have a contradiction, since this would mean that \( c \circ \ldots \circ c \) (n-times) does not give us the identity for any \( n \), but that \( \phi(c) \) says that \( c \) has finite order.

Consider \( \Delta \subset T \cup \{\phi(c)\} \cup \{c \circ \ldots \circ c \neq e : n = 1, 2, 3, \ldots\} \), and let \( n \) be such that \( \Delta \subset T \cup \{\phi(c)\} \cup \{c \circ \ldots \circ c \neq e : k < n\} \). Consider \( G = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \ldots, n - 1\} \) with \( \circ \) defined as addition mod \( n \) and \( e = 0 \). Then interpret \( c \) as 1. \( \phi(c) \) holds, since 1 has order \( n \). However, for all \( k < n \), if we look at \( c \circ \ldots \circ c \) \( k \)-times, we just get \( k \neq 0 \). Thus, \( (G, +(\text{ mod } n), 0) \models \Delta \).

Thus, \( \Gamma \) is finitely satisfiable, so by compactness, it is satisfiable, giving us our contradiction.
3. We can use a similar argument to show that there is no sentence \( \phi \) such a graph is connected if and only if it satisfies \( \phi \). Suppose for contradiction that there is such a sentence \( \phi \) in the language of graphs, \( \mathcal{L} = \{E\} \). Then let \( \sigma_0(x, y) = \neg E(x, y) \) and for \( n > 0 \), \( \sigma_n(x, y) = \forall v_1 \ldots \forall v_n \neg (E(x, v_1) \land E(v_1, v_2) \land \ldots \land E(v_n, y)) \). So if \( \sigma_n(x, y) \) is true in of some vertices \( x \) and \( y \) in a graph, then there is no path of length \( n + 2 \) between \( x \) and \( y \). Let \( c \) and \( d \) be new constants and consider the expanded language \( \mathcal{L} = \{E, c, d\} \).

Let \( T = \{\forall x \neg E(x, x), \forall x \forall y (E(x, y) \rightarrow E(y, x)), \phi\} \cup \{\sigma_n(c, d) : n \in \mathbb{N}\} \). Let \( \Delta \subset T \) be a finite subset and let \( n \in \mathbb{N} \) be such that \( \Delta \subset \{\forall x \neg E(x, x), \forall x \forall y (E(x, y) \rightarrow E(y, x)), \phi\} \) for \( n > 0 \). Interpret \( c \) as \( v_0 \) and \( d \) as \( v_{n+2} \). Then \( \mathcal{G} = \{\forall x \neg E(x, x), \forall x \forall y (E(x, y) \rightarrow E(y, x)), \phi\} \) since it is a graph and it is connected, and \( \mathcal{G} \models \sigma_i(c, d) \) for \( 1 \leq i \leq n \), since the shortest path from \( c \) to \( d \) has \( n + 1 \) vertices between them. Thus, \( \mathcal{G} \models \Delta \).

Consider a graph \( \mathcal{G} = (\{v_0, v_1, \ldots, v_{n+1}, v_{n+2}\}, E, v_0, v_{n+2}) \) on \( n + 3 \) vertices, \( v_0, v_1, \ldots, v_{n+1}, v_{n+2} \) such that \( E(v_1, v_{i+1}) \) for \( 0 \leq i < n + 2 \). Interpret \( c \) as \( v_0 \) and \( d \) as \( v_{n+2} \). Then \( \mathcal{G} \models \{\forall x \neg E(x, x), \forall x \forall y (E(x, y) \rightarrow E(y, x)), \phi\} \) since it is a graph and it is connected, and \( \mathcal{G} \models \sigma_i(c, d) \) for \( 1 \leq i \leq n \), since the shortest path from \( c \) to \( d \) has \( n + 1 \) vertices between them. Thus, \( \mathcal{G} \models \Delta \).

So by compactness, since \( T \) is finitely satisfiable, \( T \) is satisfiable by some graph \( \mathcal{M} \). However, \( c \) and \( d \) are vertices in \( \mathcal{M} \) such that there is no path between them (since paths must have finite length). So \( \mathcal{M} \) is disconnected, contradicting that \( \mathcal{M} \models \phi \).

Hence, no such \( \phi \) can exist.

4. Let \( \mathcal{L} = \{<\} \). Recall that \(<\) is an well-ordering if there are no infinitely descending chains. That is, there are no sequence of elements \( (x_i : i \in \mathbb{N}) \) such that \( x_{i+1} < x_i \). For example, \((\mathbb{N}, <)\) is a well-ordering. However, we can show that there exists \((X, <) \equiv (\mathbb{N}, <)\) such that \(<\) is not a well-ordering on \( X \). This result may seem surprising, but elementary equivalence only guarantees that the two structures satisfy the same first order sentences.

Let \( \mathcal{L}' = \{<\} \cup \{c_i : i \in \mathbb{N}\} \), the expanded language with countably many new constants and let \( T \) be the \( \mathcal{L} \)-theory \( Th(\mathbb{N}, <) \cup \{c_{i+1} < c_i : i \in \mathbb{N}\} \).

Let \( \Delta \) be a finite subset of this theory and let \( n \) be such that \( \Delta \subset Th(\mathbb{N}, <) \cup \{c_n < c_{n-1}, \ldots, c_1 < c_0\} \).

Interpret \((\mathbb{N}, <, c_i : i \in \mathbb{N})\) as an \( \mathcal{L}' \)-structure by interpreting \( c_i = n - i \) for \( 0 \leq i \leq n \) and \( c_i = i \) for \( i > n \). Then \( c_n < c_{n-1} < \ldots < c_1 < c_0 \), so \((\mathbb{N}, <, c_i : i \in \mathbb{N}) \models \Delta \).

Thus, since \( Th(\mathbb{N}, <) \cup \{c_{i+1} < c_i : i \in \mathbb{N}\} \) is finitely satisfiable, by compactness it is satisfiable by some structure \((X, <, c_i : i \in \mathbb{N})\), so if we consider \((X, <)\), this is elementarily equivalent to \((\mathbb{N}, <)\) and is not well-ordered, since it has an infinite descending chain.

5. Infinitary Ramsey Theorem Example (coming soon)

3.5 Types

A natural extension of definability is type definability, that is, sets which we can describe with infinitely many formulas. Let \( \mathcal{L} \) be a language and fix an \( \mathcal{L} \)-structure \( \mathcal{M} \). Let \( A \subset M \), and let \( Th_A(\mathcal{M}) \) be the set of all \( \mathcal{L}_A \)-sentences which are true in \( \mathcal{M} \).
Definition 42. Let $p$ be a set of $\mathcal{L}_A$-formulas in free variables $v_1, \ldots, v_n$. Call $p$ and $n$-type if $p \cup \text{Th}_A(M)$ is satisfiable. We say that $p$ is a complete $n$-type if for every $\phi(\overline{v})$, either $\phi(\overline{v}) \in p$ or $\neg \phi(\overline{v}) \in p$. We let $S_n^M(A)$ denote the set of all complete $n$-types.

Examples:

1. Let $\mathcal{M} = (\mathbb{Q}, <)$. Let $A = \mathbb{N}$ and let $p(v)$ be the type \{1 < v, 2 < v, 3 < v, \ldots\}. Then $p$ is a 1-type over $A$, since we can show by compactness that $p(v) \cup \text{Th}_A(\mathcal{M})$ is satisfiable.

2. Given $a \in M$, $tp(a/A)$ is the set of $\mathcal{L}_A$-formulas $\phi(v)$ which are true of $a$.

Fun fact: the compactness theorem coincides with topological compactness when you put the right topology on the set of types. (Coming soon)