This article appeared in College Mathematics Journal **33** (2002), no. 5, 406-408.

Two Quick Combinatorial Proofs of $\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2$.

Arthur T. Benjamin (benjamin@math.hmc.edu) Harvey Mudd College, Claremont, CA 91711-5590
Michael E. Orrison (orrison@math.hmc.edu) Harvey Mudd College, Claremont, CA 91711-5590

In many discrete mathematics classes, the identity $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ is a standard exercise in mathematical induction. Alternative proofs are possible that allow this identity to be appreicated from different perspectives. For instance, in [2], seven different geometric proofs are presented.

However, since $\frac{n^2(n+1)^2}{4}$ is equal to $\binom{n+1}{2}^2$, it seems only natural that a simple combinatorial proof should be possible. We present two such proofs. Specifically, we find sets S and T where $|S| = \sum_{k=1}^{n} k^3$ and $|T| = \binom{n+1}{2}^2$, then exhibit a bijection (i.e., a one-to-one, onto function) between them.

Let S denote the set of 4-tuples of integers from 0 to n whose last component is strictly bigger than the others; that is,

$$S = \{(h, i, j, k) \mid 0 \leq h, i, j < k \leq n\}.$$

For $1 \leq k \leq n$, there are k^3 ways to choose h, i, j given the last component k. Hence, $|S| = \sum_{k=1}^{n} k^3$.

Let T denote the set of ordered pairs of two element subsets of $\{0, \ldots n\}$, which may be expressed as

$$T = \{((x_1, x_2), (x_3, x_4)) \mid 0 \leq x_1 < x_2 \leq n, \;\; 0 \leq x_3 < x_4 \leq n\}.$$

Clearly $|T| = \binom{n+1}{2}^2$.

To see that S and T have the same size, we find a bijection $f : S \to T$ between these sets. Specifically,

$$f((h, i, j, k)) = \begin{cases} ((h, i), (j, k)), & \text{if } h < i \\ ((j, k), (i, h)), & \text{if } h > i \\ ((i, k), (j, k)), & \text{if } h = i \end{cases}$$

is a bijection since the cases $h < i$, $h > i$, and $h = i$ are mapped onto ordered pairs $((x_1, x_2), (x_3, x_4))$ where $x_2 < x_4$, $x_2 > x_4$, and $x_2 = x_4$, respectively. Thus, $|S| = |T|$.

A simpler correspondence arises when we interpret $\binom{n+1}{2}$ as the number of ways to choose two elements from $\{1, \ldots, n\}$ with repetition allowed. This time we let

$$S = \{(h, i, j, k) \mid 1 \leq h, i, j \leq k \leq n\},$$

which has size $|S| = \sum_{k=1}^{n} k^3$, and let

$$T = \{((x_1, x_2), (x_3, x_4)) \mid 1 \leq x_1 \leq x_2 \leq n, \quad 1 \leq x_3 \leq x_4 \leq n\},$$

which has size $\binom{n+1}{2}^2$. Here, our bijection $g : S \to T$ has just two cases:

$$g((h, i, j, k)) = \begin{cases} ((h, i), (j, k)) & \text{if } h \leq i \\ ((j, k), (i, h-1)) & \text{if } h > i \end{cases}.$$

The first case maps onto those $((x_1, x_2), (x_3, x_4))$ where $x_2 \leq x_4$, and the second case maps onto those where $x_2 > x_4$. Hence g is a bijection, and $|S| = |T|$.

Another combinatorial approach to this identity is utilized in [1] and [3] using the set S from our first proof. By conditioning on the number of 4-tuples in S with $2, 3$ and 4 distinct elements, it follows that $\sum_{k=1}^{n} k^3 = \binom{n+1}{2} + \binom{n+1}{3}6 + \binom{n+1}{4}3!$, which algebraically simplifies to $\frac{n^2(n+1)^2}{4}$. Our motivation in this note was to avoid the use of algebra and arrive at $\binom{n+1}{2}^2$ in a purely combinatorial way.

We leave the reader with the challenge of finding a combinatorial proof of

$$\sum_{k=1}^{n} k^2 = \frac{1}{4}\binom{2n+2}{3}.$$

References

[1] George Mackiw, A Combinatorial Approach to Sums of Integer Powers, *Mathematics Magazine* **73** (2000) 44–46.

[2] Roger B. Nelsen, *Proofs Without Words*, MAA, Washington DC, 1993.

[3] Marta Sved, Counting and Recounting, *The Mathematical Intelligencer* **5** (1983) 21–26.