Two Quick Combinatorial Proofs of $\sum_{k=1}^{n} k^3 = \left(\frac{n+1}{2}\right)^2$.

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In many discrete mathematics classes, the identity $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ is a standard exercise in mathematical induction. Alternative proofs are possible that allow this identity to be appreciated from different perspectives. For instance, in [2], seven different geometric proofs are presented.

However, since $\frac{n^2(n+1)^2}{4}$ is equal to $\left(\frac{n+1}{2}\right)^2$, it seems only natural that a simple combinatorial proof should be possible. We present two such proofs. Specifically, we find sets $S$ and $T$ where $|S| = \sum_{k=1}^{n} k^3$ and $|T| = \left(\frac{n+1}{2}\right)^2$, then exhibit a bijection (i.e., a one-to-one, onto function) between them.

Let $S$ denote the set of 4-tuples of integers from 0 to $n$ whose last component is strictly bigger than the others; that is,

$$S = \{(h, i, j, k) \mid 0 \leq h, i, j < k \leq n\}.$$

For $1 \leq k \leq n$, there are $k^3$ ways to choose $h, i, j$ given the last component $k$. Hence, $|S| = \sum_{k=1}^{n} k^3$.

Let $T$ denote the set of ordered pairs of two element subsets of $\{0, \ldots, n\}$, which may be expressed as

$$T = \{((x_1, x_2), (x_3, x_4)) \mid 0 \leq x_1 < x_2 \leq n, \ 0 \leq x_3 < x_4 \leq n\}.$$

Clearly $|T| = \left(\frac{n+1}{2}\right)^2$.

To see that $S$ and $T$ have the same size, we find a bijection $f : S \rightarrow T$ between these sets. Specifically,

$$f((h, i, j, k)) = \begin{cases} 
((h, i), (j, k)), & \text{if } h < i \\
((j, k), (i, h)), & \text{if } h > i \\
((i, k), (j, k)), & \text{if } h = i
\end{cases}$$

is a bijection since the cases $h < i$, $h > i$, and $h = i$ are mapped onto ordered pairs $((x_1, x_2), (x_3, x_4))$ where $x_2 < x_4$, $x_2 > x_4$, and $x_2 = x_4$, respectively. Thus, $|S| = |T|$. 

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A simpler correspondence arises when we interpret \( \binom{n+1}{2} \) as the number of ways to choose two elements from \( \{1, \ldots, n\} \) with repetition allowed. This time we let 

\[ S = \{(h, i, j, k) \mid 1 \leq h, i, j \leq k \leq n\}, \]

which has size \( |S| = \sum_{k=1}^{n} k^3 \), and let 

\[ T = \{((x_1, x_2), (x_3, x_4)) \mid 1 \leq x_1 \leq x_2 \leq n, \ 1 \leq x_3 \leq x_4 \leq n\}, \]

which has size \( \binom{n+1}{2}^2 \). Here, our bijection \( g : S \rightarrow T \) has just two cases:

\[
g((h, i, j, k)) = \begin{cases} 
((h, i), (j, k)) & \text{if } h \leq i \\
((j, k), (i, h-1)) & \text{if } h > i
\end{cases}
\]

The first case maps onto those \( ((x_1, x_2), (x_3, x_4)) \) where \( x_2 \leq x_4 \), and the second case maps onto those where \( x_2 > x_4 \). Hence \( g \) is a bijection, and \( |S| = |T| \).

Another combinatorial approach to this identity is utilized in [1] and [3] using the set \( S \) from our first proof. By conditioning on the number of 4-tuples in \( S \) with 2, 3 and 4 distinct elements, it follows that 

\[
\sum_{k=1}^{n} k^3 = \binom{n+1}{2} + \binom{n+1}{3}6 + \binom{n+1}{4}3!,
\]

which algebraically simplifies to \( \frac{n^2(n+1)^2}{4} \). Our motivation in this note was to avoid the use of algebra and arrive at \( \binom{n+1}{2}^2 \) in a purely combinatorial way.

We leave the reader with the challenge of finding a combinatorial proof of

\[
\sum_{k=1}^{n} k^2 = \frac{1}{4} \binom{2n + 2}{3}.
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References

