Section 1.4: Directed Graphs

Definitions and Examples, Vertex Degrees, Eulerian Digraphs, Orientations and Tournaments
Eulerian Digraphs

Definition (1.4.22). A digraph is Eulerian if it has an Eulerian circuit.

Lemma (1.4.23). If $G$ is a digraph with $\delta^+(G) \geq 1$, then $G$ contains a cycle. The same conclusion holds when $\delta^-(G) \geq 1$.

Theorem (1.4.24). A digraph is Eulerian if and only if $d^+(v) = d^-(v)$ for each vertex $v$ and the underlying graph has at most one component.

Proof. (sketch) Necessity is clear, so we need only show sufficiency. To do this, first prove that every non-extendible trail in the graph is closed. Then show that a trail of maximal length must be an Eulerian circuit.
de Bruijn Graph

Is there a cyclic arrangement of $2^n$ binary digits such that the $2^n$ strings of $n$ consecutive digits are all distinct?

Example. When $n = 4$, the following sequence works:

$$(0000111101100101).$$

Let $D_n$ be the digraph whose vertices are the binary $(n−1)$-tuples and where there is an edge from $a$ to $b$ if the last $n−2$ entries of $a$ agree with the first $n−2$ entries of $b$.

To help us construct the cyclic arrangement described above, we label each edge from $a$ to $b$ with the last entry of $b$. 
Theorem (1.4.26). The digraph $D_n$ is Eulerian. Furthermore, the edge labels on the edges in any Eulerian circuit of $D_n$ form a cyclic arrangement in which the $2^n$ consecutive segments of length $n$ are distinct.

Proof. (sketch) First show that every vertex has indegree 2 and outdegree 2. Then show that $D_n$ is strongly connected. Theorem 1.4.24 then implies that $D_n$ is Eulerian.

For the second statement, let $C$ be an Eulerian circuit of $D_n$. Then show that each of the $2^n$ consecutive segments of length $n$ described above correspond uniquely to an edge of $D_n$ as we traverse $C$. Note that each segment $a_1, \ldots, a_n$ corresponds to a vertex $(a_1, \ldots, a_{n-1})$ and the edge labeled $a_n$ that is leaving that vertex.
Orientations and Tournaments

**Definition (1.4.27).** An orientation of a graph $G$ is a digraph $D$ obtained from $G$ by choosing an orientation for each edge $xy \in E(G)$. An oriented graph is an orientation of a simple graph. A tournament is an orientation of a complete graph.

**Definition (1.4.29).** In a digraph, a king is a vertex from which every vertex is reachable by a path of length at most 2.

**Proposition (1.4.30).** (Landau [1953]) Every tournament has a king.

*Proof.* Let $x$ be a vertex of maximum outdegree in the tournament $T$, and suppose $x$ is not a king. Then there exists a vertex $y$ that is not reachable from $x$ by a path of length at most 2. Every successor of $x$ must therefore be a successor of $y$, implying that $d^+(y) > d^+(x)$ (since $y \to x$). But this contradicts the fact that $x$ has maximum outdegree. Thus, $x$ must be a king. \qed
Exercises

1.4.7. (–) Prove or disprove: If $D$ is an orientation of a simple graph with 10 vertices, then the vertices of $D$ cannot have distinct outdegrees.

1.4.8. (–) Prove that there is an $n$-vertex tournament with indegree equal to outdegree at every vertex if and only if $n$ is odd.

1.4.36(a-b). Let $T$ be a tournament having no vertex with indegree 0. Prove that if $x$ is a king in $T$, then $T$ has another king in $N^-(x)$. Use this result to then show that $T$ has at least three kings.