Section 6.5: The Kernel and Range of a Linear Transformation
In this section, we extend the important notions of null space, column space, rank, and nullity for matrix-vector multiplication to the general setting of linear transformations.

**Definition**

Let $T : V \rightarrow W$ be a linear transformation. The **kernel** of $T$ is

$$\ker(T) = \{ v \in V \mid T(v) = 0 \}.$$  

The **range** of $T$ is

$$\text{range}(T) = \{ T(v) \mid v \in V \}.$$  

**Example**

**Example**
Theorem 6.18
Let \( T : V \rightarrow W \) be a linear transformation.

1. The kernel of \( T \) is a subspace of \( V \).
2. The range of \( T \) is a subspace of \( W \).

Definition
Let \( T : V \rightarrow W \) be a linear transformation. The \textbf{rank} of \( T \), which we denote by \( \text{rank}(T) \), is the dimension of the range of \( T \). The \textbf{nullity} of \( T \), which we denote by \( \text{nullity}(T) \), is the dimension of the kernel of \( T \).
Theorem 6.19 (The Rank Theorem)

Let $V$ be a finite dimensional vector space, and let $T : V \to W$ be a linear transformation. Then

$$\text{rank}(T) + \text{nullity}(T) = \dim V.$$
Definition (one-to-one, onto)

A linear transformation \( T : V \rightarrow W \) is **one-to-one** if it maps distinct vectors in \( V \) to distinct vectors in \( W \). If \( \text{range}(T) = W \), then we say \( T \) is **onto**.

**Question**

Can you find examples of linear transformations that are (1) one-to-one but not onto, (2) onto but not one-to-one, (3) one-to-one and onto, and (4) neither one-to-one nor onto?

**Answer**
Theorem 6.20
A linear transformation $T : V \to W$ is one-to-one if and only if $\ker(T) = \{0\}$.

Theorem 6.21
Let $V$ and $W$ be finite-dimensional vector spaces such that $\dim V = \dim W$. Then a linear transformation $T : V \to W$ is one-to-one if and only if it is onto.
**Theorem 6.22**

Let $T : V \to W$ be a one-to-one linear transformation. If $S = \{v_1, \ldots, v_k\}$ is a linearly independent subset of $V$, then $T(S) = \{T(v_1), \ldots, T(v_k)\}$ is a linearly independent subset of $W$.

**Corollary 6.23**

Let $\dim V = \dim W = n$. Then a one-to-one linear transformation $T : V \to W$ maps a basis of $V$ to a basis of $W$. 
Theorem 6.24
A linear transformation $T : V \to W$ is invertible if and only if it is one-to-one and onto.

Definition (isomorphism)
A linear transformation $T : V \to W$ is called an isomorphism if it is one-to-one and onto. If there exists an isomorphism from $V$ to $W$, we say that $V$ is isomorphic to $W$ and write $V \cong W$.

Theorem 6.25
Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.