Realizing the 2-Associahedron

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Abstract

The associahedron has appeared in numerous contexts throughout the field of mathematics. By representing the associahedron as a poset of tubings, Michael Carr and Satyan L. Devadoss were able to create a generalized version of the associahedron in the graph-associahedron. We seek to create an alternative generalization of the associahedron by considering a particle-collision model. By extending this model to what we dub the 2-associahedron, we seek to further understand the space of generalizations of the associahedron.

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Part I

Background Material

Chapter 1

The Associahedron

In this chapter, we seek to develop an understanding of various models for the associahedron. To define the associahedron, we look to the space of all possible collisions of particles on a line, which can be visualized as bracketings of a path. From there, we provide an alternative model of the associahedron through tubings and discuss the graph-associahedron which arises from this representation.

1.1 Constructing the Associahedron through Particle Collisions

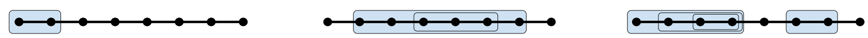
In order to study the space of particles colliding on a line, we must first define a model for collision.

Definition 1.1. Let $S = \{p_1, p_2, \ldots p_n\}$ be a set of n particles on a line, indexed by their order on the line. A *simple collision* is a proper subset of S containing $k > 1$ consecutive particles, $\{p_i, p_{i+1}, \ldots, p_{k+i-1}\}$. We say two simple collisions b_1 and b_2 *intersect* if $b_1 \cap b_2 \neq \emptyset$, $b_1 \not\subset b_2$, and $b_2 \not\subset b_1$.

Simple collisions are *compatible* if they do not intersect. A *collision* on S is a set of simple collisions on S such that every pair of elements in S are compatible. Various collisions can be seen in Figure 1.1.

From this definition of collisions, we may now construct the associahedron.

Definition 1.2. Let $\mathfrak{A}(n)$ be the poset of collisions on n particles, where $c \prec c'$ if c is obtained from c' by adding simple collisions. The *associahedron* K_n is the convex polytope whose face poset is isomorphic to $\mathfrak{A}(n)$.



a. A simple collision of p_1 and p_2 **b.** Two nested simple collisions **c.** A collision composed of four simple collisions

Figure 1.1 Visualizations of various collisions of seven particles.

This model provides a simple definition for the associahedron, which can be visualized through "bracketings" on a path, as shown in Figure 1.1. The associahedron K_4, with faces labeled with corresponding bracketings, is shown in Figure 1.2a.

1.2 Tubings and the Graph-Associahedron

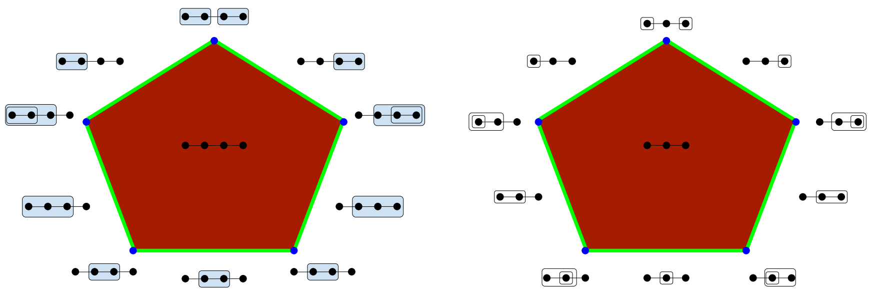
By depicting particle collisions as paths, we find a realization of the associahedron on specific graphs. However, this idea of bracketing cannot be generalized to arbitrary simple graphs as one would hope. For such a generalization, we look to the idea of *tubings*, as developed by Michael Carr and Satyan L. Devadoss.

Definition 1.3. Let G be a graph. A *tube* is a proper nonempty subset of the vertices of G whose induced subgraph is connected. Two tubes t_1 and t_2 may interact in three distinct ways on the graph.

1. Tubes are *nested* if $t_1 \subset t_2$ or $t_2 \subset t_1$.
2. Tubes *intersect* if they are not nested and $t_1 \cap t_2 \neq \emptyset$.
3. Tubes are *adjacent* if they do not intersect and $t_1 \cup t_2$ is a tube.

Two tubes are *compatible* if they do not intersect and they are not adjacent. A *tubing* T of G is a set of tubes of G such that every pair of tubes in T are compatible. [2]

This idea of tubings permits a similar poset structure to that of bracketings where, for two tubings T and T', we say $T \prec T'$ if T can be constructed by adding tubes to T'. In Figure 1.2b, we see that the poset of tubings on 3 vertices is precisely K_4. In fact, this holds in general.



a **b**

Figure 1.2 Labelings of K_4 with (a) bracketings and (b) tubings.

Proposition 1.4. *Let G be a path with $n-1$ vertices. The face poset of K_n is isomorphic to the poset of tubings on G.* [2]

The proof of the proposition follows trivially from the natural bijection between tubes and simple collisions.

Tubings thus encompass the structure of the associahedron when defined on paths, but are also well-defined on any simple graph. As such, tubings provide a natural extension of the associahedron. In fact, Devadoss and Carr proved that this extention creates quite well-behaved objects.

Proposition 1.5. *Let G be a graph. The poset of tubings on G is isomorphic to the face poset of some polytope P. We call P the graph-associahedron of G.* [2]

Chapter 2

The 2-Associahedron

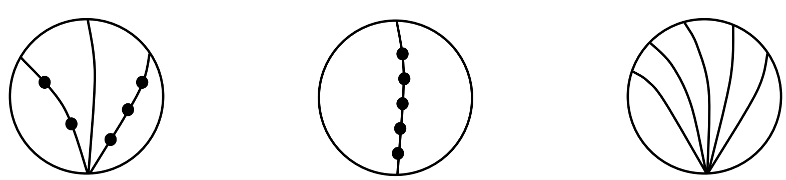
From Proposition 1.5, we see that the associahedron is merely a specific instance of some larger space of objects. By generalizing the tubing model of the assoicahedron, Devadoss and Carr were able to create another object in the graph-associahedron.

A natural progression is to thus generalize other representations of the associahedron in order to develop a better picture of this larger space. One alternative is the 2-Associahedron, which arises from expanding the particle collision model to two dimensions.

2.1 Quilted Spheres

To describe collisions to two dimensions, we utilize the idea of a quilted sphere.

Definition 2.1. A *quilted sphere*, denoted $Q^k_{n_1,\ldots,n_k}$ is composed of a sphere with k *seams*, all of which meet at the south pole. The i-th seam holds n_i



a. A sketch of $Q^3_{2,0,3}$ **b.** A sketch of Q^1_5 **c.** A sketch of $Q^5_{0,0,0,0,0}$

Figure 2.1 Examples of various quilted spheres.

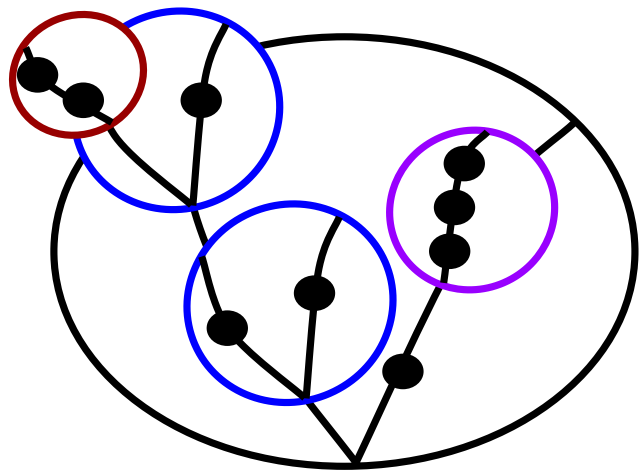


Figure 2.2 A configuration on $Q^3_{3,2,4}$. Bubbles of the same color are similar, bubbles of different colors are not.

particles. Examples of quilted spheres can be found in Figure 2.1. A quilted sphere is said to be *stable* if it has at least one particle, at least one seam, and is not Q^1_1.

Now that we have the idea of a quilted sphere, we can begin to describe particle collisions on these objects. To model a collision, we create *bubbles* containing particles that have collided.

Definition 2.2. A *bubble* is a quilted sphere whose base point lies on the seam of another quilted sphere. A *marked point* is either a particle or the base point of a bubble.

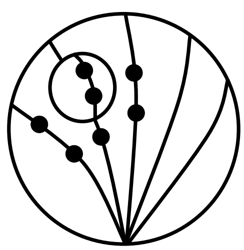
We say two bubbles are *similar* if they contain precisely the same seams of the root sphere. Figure 2.2 provides examples of similar bubbles.

Definition 2.3. A *type 1* move occurs when two or more consecutive marked points on a single seam collide. These particles bubble off into a single-seam bubble, as shown in Figure 2.3a.

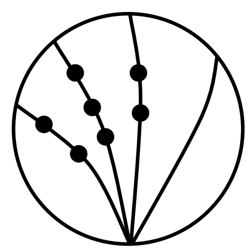
A *type 2* move occurs when $j \geq 2$ consecutive seams collide on a bubble. These seams fuse into a single seam in one of two ways.

1. If none of the j seams contained any particles, the new seam is also empty, as shown in Figure 2.3b.

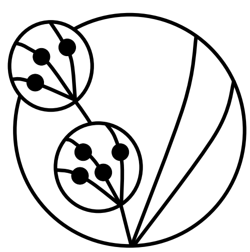
2. Otherwise, some $k \geq 1$ bubbles form. The particles which were on the colliding seams are now distributed over the bubbles. If a particle started on the i-th seam of the collision, it must end on the i-th seam of a bubble. Two examples of such moves are shown in Figures 2.3c and 2.3d. If the seams involved in this collision are a proper subset of the seams on the bubble, the same seam collision must occur on all similar bubbles.



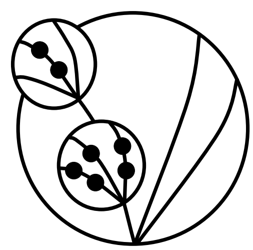
a. A type 1 move.



b. A type 2 move of the first kind.



c. A type 2 move of the second kind.



d. Another type 2 move of the second kind.

Figure 2.3 Various moves on $Q^5_{2,3,2,0,0}$.

Definition 2.4. We denote $P^k_{n_1,\ldots,n_k}$ as the set of all allowable configurations of $Q^k_{n_1,\ldots,n_k}$ under these moves. Similarly to one dimensional collisions, we may create a well-defined poset from $P^k_{n_1,\ldots n_k}$ where $M \prec M'$ if M can be made from M' through the above moves.

This poset structure strongly mimics that of one dimensional particle collisions, which gave rise to the associahedron. In fact, we may recover the associahedron from quilted spheres in two distinct ways. The first is through single-seamed spheres.

Theorem 2.1. *The face poset of the associahedron K_n is isomorphic to P^1_n.*

The proof of Theorem 2.1 follows from the trivial bijection between type 1 moves on P^1_n and n particles colliding on a line. Through a similarly trivial bijection from type 2 moves on single-particle spheres, we can recover the associahedron in another way.

Theorem 2.2. *The fact poset of the associahedron K_n is isomorphic to $P^n_{0,0,\ldots,0,1,0,\ldots,0}$.*

Through Theorems 2.1 and 2.2, the space of quilted spheres is clearly a generalization of the associahedron. We thus seek to study these spaces to further understand the shape of associativity.

Part II

Analyzing the 2-Associahedron

Chapter 3

Constructing the 2-Associahedron through Truncation

In general, the graph-associahedron can be constructed through a series of truncations on the configuration spaces of non-nested tubings. By applying this idea to quilted spheres, we can see the 2-associahedron as truncations of the configuration space of non-nested bubblings on a quilted sphere.

3.1 The Graph-Associahedron as Truncations

To construct the graph-associahedron through truncation, we begin by defining a *configuration space* on a graph G.

Definition 3.1. The *configuration space* on a graph G is a polytope whose faces are labeled with the non-nested tubings on G. For faces F and F', $F \subset F'$ if the tubing of F' can be nested within the tubing of F. An example configuration space is shown in Figure 3.1.

For any graph, the configuration space is in fact quite simple.

Theorem 3.1. *For a graph G on n vertices, the configuration space is precisely the $(n-1)$-simplex.*

With this idea of a configuration space, we can now construct the graph-associahedron quite simply.

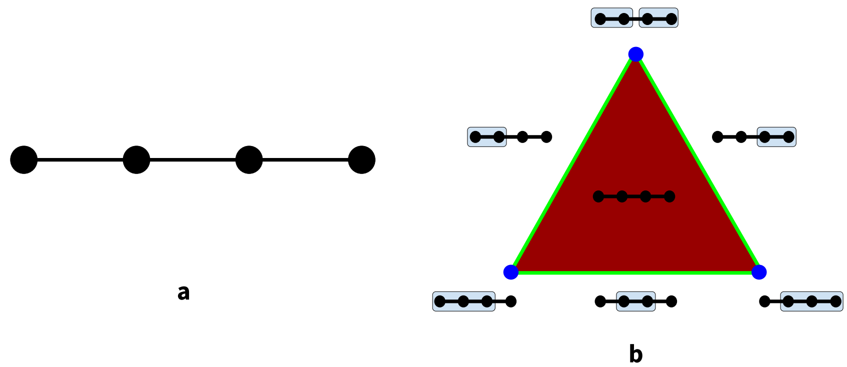


Figure 3.1 A graph (a) and its associated configuration space (b).

Theorem 3.2. *The graph-associahedron P on a graph G is precisely the polytope attained by truncating all faces of the configuration space on G which are labeled with* 1*-tubings, in increasing order of dimension. An example truncation is shown in Figure* 3.2*.*

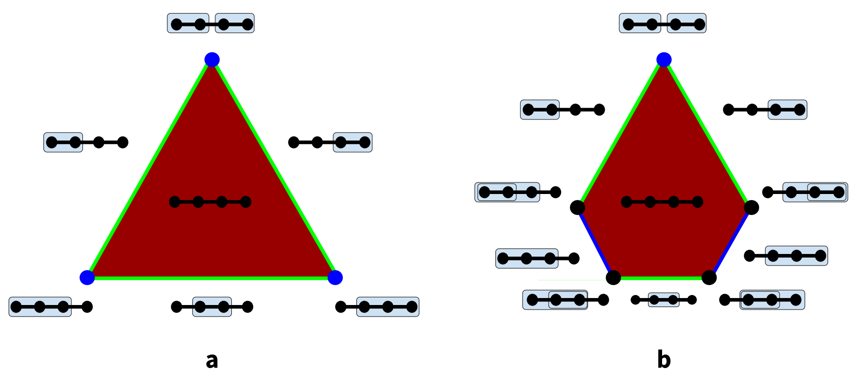


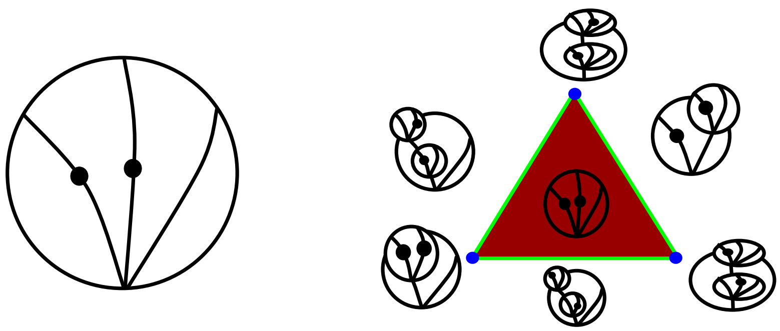
Figure 3.2 A configuration space (a) and its truncation into a associahedron (b).

This relationship is paramount to the proof that the associahedron itself is always a polytope. Because it is constructed by truncating a simplex in such a clean manner, the resultant associahedron is necessarily a polytope.

3.2 The 2-Associahedron as Truncations

Similarly to configuration spaces on graphs, we can define a configuration space on quilted spheres.

Definition 3.2. We define the *configuration space* on a quilted sphere Q as the poset of non-nested bubblings on Q. An example configuration space is shown in Figure 3.3.



a **b**

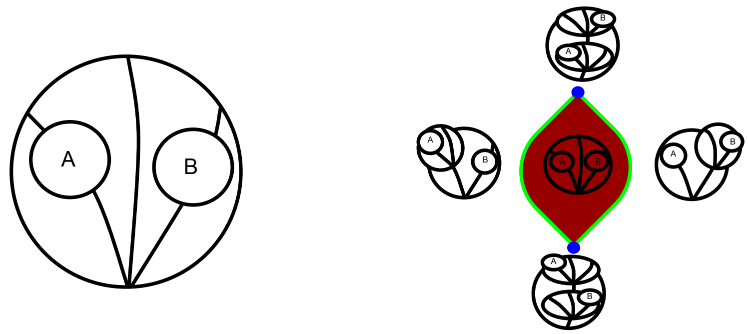
Figure 3.3 The sphere $Q^3_{1,1,0}$ (a) and its configuration space (b).

Whereas the configuration spaces for the graph-associahedron are well-behaved polytopes, quilted spheres are not so simple.

Theorem 3.3. *There exist quilted spheres whose configuration space is not a polytope. Specifically, any quilted sphere with an empty seam between two non-empty seams does not have a polytopal configuration space.*

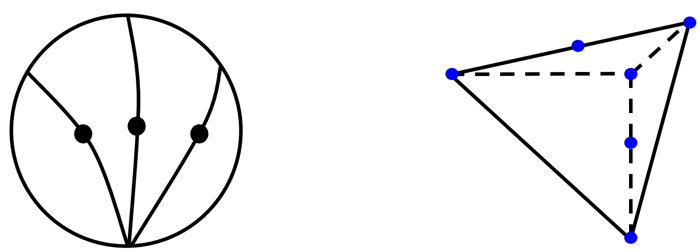
Proof. The proof of Theorem 3.3 is rather straight forward. For any quilted sphere with an empty seam between two non-empty seams, there is an element F of the configuration space of the form depicted in Figure 3.4a. By basic inspection, it can be seen that F is a two-dimensional face, as seen in Figure 3.4b. Thus, this configuration space must have a bigonal face, and thus cannot be a polytope. □

Theorem 3.3 provides us with numerous examples of quilted spheres with "bad" configuration spaces, but it is in no way exhaustive. One example which does not conform to Theorem 3.3 arises from $Q^3_{1,1,1}$ as seen in Figure 3.5



a **b**

Figure 3.4 An element of a configuration space (a) and its associated face (b).



a **b**

Figure 3.5 The sphere $Q^3_{1,1,1}$ (a) and its configuration space (b), a 3-simplex with two additional vertices.

Now that we have a basic understanding of the configuration space of a quilted sphere, we can attempt to draw parallels to configuration spaces of the graph-associahedron. Specifically, we hope to construct the 2-associahedron from its configuration space.

Conjecture 3.3. *The* 2*-associahedron is the result of truncating the associated configuration space in increasing order of dimension.*

In practice, this method of truncation has always worked. However, we have not been able to prove its validity. If Conjecture 3.3 can be proven true, we would be one step closer to fully understanding the 2-associahedron.

Chapter 4

The Constrainahedron

One of the biggest difficulties in dissecting the 2-associahedron is the natural asymmetry of the object: particle collisions and seam collisions behave in completely different manners. Particle collisions precisely model the regular associahedron, and are thus rather well-understood. Seam collisions, however, are much less understood. To gain a better understanding of the nature of seam collisions, we define simpler object dominated primarily by seam interactions. We call this object the *constrainahedron*.

4.1 Defining the Constrainahedron

To generate the constrainahedron, we begin with the notion of a *constrained quilted sphere*.

Definition 4.1. A *constrained quilted sphere* $CQ_{k,n}$ is a quilted sphere with k seams and some n particles on each seam. This sphere is constrained such that, when any particle moves vertically on its respective seam, each particle in the same row must move identically. A few examples are shown in Figure 4.1. Again, we can define a poset on these constrained quilted spheres such that $S \prec S'$ if S can be achieved from S' through additional collisions. [1]

Through this poset, we can realize the constrainahedron.

Definition 4.2. The *constrainahedron* $C_{k,n}$ is the object with face poset equivalent to the poset of moves on $CQ_{k,n}$.

In general, we have struggled to achieve a concise description of the legal moves on the constrainahedron. However, for small numbers of seams, we

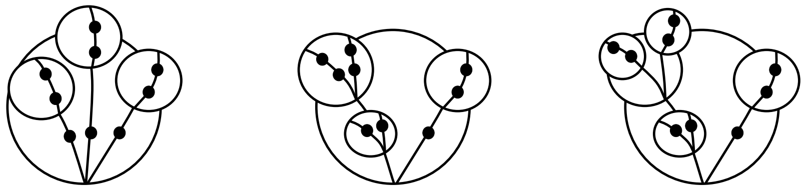


Figure 4.1 Various valid configurations on $CQ_{3,3}$.

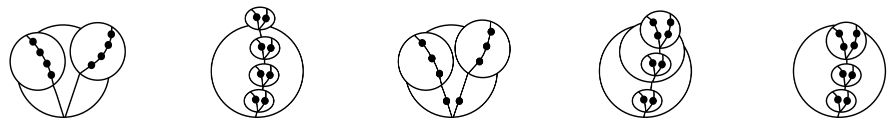
can describe the rules. When $k = 1$, the constrainahedron mirrors the associahedron precisely and the rules are trivially those of the associahedron. When $k = 2$, the constrainahedron can be described through four distinct moves. To classify these moves, we first begin with some terminology.

Definition 4.3. On a configuration of $C_{2,n}$, we define an *outer bubbble* as any single-seam bubble which does not lie inside a 2-seam bubble. Similarly, we define an *inner bubble* as a single-seam bubble which does lie on some 2-seam bubble.

Definition 4.4. For the constrainahedron $C_{2,n}$, we can classify all legal moves into four categories.

1. (a) All marked points on a 2-seam bubble collide.
   (b) The seams of a 2-seam bubble with m marked points collide, forming m distinct bubbles.
2. Some $j \geq 2$ rows of marked points collide on either a 2-seamed bubble or an inner bubble. This collision does not involve *all* marked points on this bubble.
3. Some $j \geq 2$ marked points collide on an outer bubble.
4. The seams of a 2-seam bubble fuse, with m marked points on each seam. Fewer than m new bubbles form.

Examples of these moves can be seen in Figure 4.2



a. A type 1a move. **b.** A type 1b move. **c.** A type 2 move. **d.** A type 3 move. **e.** A type 4 move.

Figure 4.2 Various moves on $C_{2,4}$.

4.2 Results from the Constrainahedron

Now that we have a general picture of the constrainahedron, we can begin to analyze the result of this construction. As mentioned previously, the constainahedron in its simplest form precisely recovers the associahedron.

Theorem 4.1. *The constrainahedra $C_{1,n}$ and $C_{n,1}$ are both isomorphic to the associahedron K_n.*

The proof of Theorem 4.1 follows directly from analysis of the legal moves on $C_{1,n}$ and $C_{n,1}$, as they mirror precisely the moves on the associahedron.

In the simplest case, the constrainahedron mirrors a known associative shape, so a natural question to ask is whether the constrainahedron generates other known associative structures. In fact, one can see that the multiplihedron can be recovered from the constrainahedron.

Theorem 4.2. *The constrainahedron $C_{2,n}$ is isomorphic to the multiplihedron on a path of length $n-1$.*

Before we prove Theorem 4.2, we must define the multiplihedron. In a paper by Satyan L. Devadass and Stefan Forcey, they provide a construction for the multiplihedron through broken, thin, and thick tubes on a graph.

Definition 4.5. To construct the multiplihedron, we begin with a path on n vertices which is fully encompassed in a broken tube. We then consider the poset generated by the following moves.

1. (a) A broken tube becomes a thin tube.
   (b) A broken tube becomes a thick tube.
2. A thin tube is added inside either a thin tube or a broken tube.

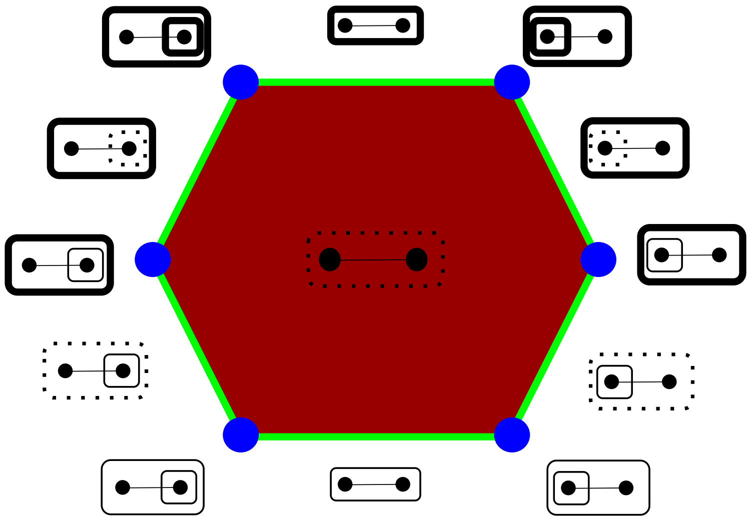


Figure 4.3 The labeled graph multiplihedron on a path of length 3.

3. A thick tube is added inside a thick tube.

4. A collection of compatible broken tubes $\{u_1, \ldots, u_n\}$ is added simultaneously inside a broken tube v and v becomes thick. For each u_i, there is no tube between u_i and v. [3]

An example labeled multiplihedron can be seen in Figure 4.3.

Proof. Now that we have the rules for constructing the multiplihedron, we can consider the proof of Theorem 4.1. The proof relies on a bijection between the tubing picture for the multiplihedron and the associated constrained quilted sphere. Considering $C_{2,n}$, let p_i denote the ith pair of marked points, and let v_i denote the ith vertex of a path of length $n-1$.

- For any two-seamed bubble containing pairs p_i through p_j, draw a thick tube around v_i through v_{j-1}.

- For any inner bubble containing pairs p_i through p_j, draw a thin tube around v_i thorugh v_{j-1}.

- For any two-seamed bubble containing pairs p_i through p_j, draw a broken tube around v_i through v_{j-1} if no other tube has already been drawn around these vertices.

This provides a bijection between constrained spheres and the multiplihedron tubings, so what remains is to demonstrate that the posets are isomorphic. This isomorphism follows from the trivial bijection between the rules suggested by their enumeration. For example, a type 2 move on the constrained quilted sphere is precisely a type 2 move on the multiplihedron. □

Part III

Future Work

Chapter 5

Future Pursuits

In this thesis, we have begun to investigate the 2-associahedron through two distinct avenues. We have considered the 2-associahedron as a construction from its configuration space and we have considered the constrainahedron, a close relative of the 2-associahedron. Neither path has been fully explored, so here we discuss possible directions to guide further inquiry.

5.1 Future Work in Configuration Spaces

As discussed in Conjecture 3.3, the biggest remaining question in this area whether the 2-associahedron always arises from an appropriate truncation of its configuration space. In pursuit of this answer, the biggest unknown precisely which quilted spheres have polytopes as their configuration spaces. In this paper, we have provided examples of spheres with configuration spaces that are not polytopes, but we have not discovered necessary and sufficient conditions for a non-polytopal configuration space to arise.

In addition to studying the 2-associahedron's configuration space, one might also apply this idea to the constrainahedron. Based off of cursory inquiry, the configuration space of the constrainahedron seems to once again be the approrpiately dimensioned simplex.

Conjecture 5.1. *The configuration space of* $C_{k,n}$ *is the* $(k+n-2)$*-simplex.*

If conjecture 5.1 holds, which we fully expect, then the natural question is how or if one could construct $C_{k,n}$ from the appropriate simplex. As $C_{2,n}$ is the multiplihedron, we know this construction cannot be as simple as the truncations of the associahedron because the multiplihedron is not, in general, a simple polytope. However, there may still exist a more complex construction which offers insight into the underlying structure of the constrainahedron.

5.2 Future Work on the Constrainahedron

The constrainahedron also permits further investigation beyond its configuration space. The most important next step is to generate a comprehensive description of the rules for an arbitrary $C_{k,n}$, as this will allow for more thorough analysis in the future. However, barring this discovery, there is still work to be done. One conjecture we have yet to prove is a basic symmetry.

Conjecture 5.2. *The constrainahedra $C_{k,n}$ and $C_{n,k}$ are isomorphic.*

The validity of Conjecture 5.2 seems apparent from the general behavior of the constrainahedron, and we expect this to follow immediately from a description of the generalized rules.

Another possible route of investigation is to extend the tubing definition of the multilpihedron to describe constrainahedra with more than two seams. For $C_{2,n}$, there is a strict hierarchy of bubbles where one always sees outer bubbles, then 2-seamed bubbles, then inner bubbles, and the multiplihedron tube construction takes advantage of this by using three distinct types of tubes with the same hierarchy restriction.

In the constrainahedron $C_{3,n}$, we see a similar hierarchy, but with outer bubbles, 3-seamed bubbles, 2-seamed bubbles, then inner bubbles. As such, we could likely define a tubing picture which is isomorphic to $C_{3,n}$ by using four types of tubes: thick, broken, colored, and thin, where colored tubes are one of two colors depending on which pair of seams is involved in the associated 2-seam bubbles. If we can transform $C_{3,n}$ into such a tubing picture, this method may begin to offer insight into a more general bijection between $C_{k,n}$ and tubes on $n-1$ vertices with $k+1$ types of tubes.

5.3 Guiding Questions

Through the lens of configuration spaces and investigations on the constrainahedron, we have begun to understand the 2-associahedron. We have seen that it is not necessarily generated from a simplex, which is distinct from the other associative objects we have seen. Through our studies of the constrainahedron, we also now know that the 2-associahedron may be a close cousin of the multiplihedron. These are valuable insights, but many questions still remain which should guide future investigation.

First and foremost, we wish to determine whether or not the 2-associahedron is always a polytope. The multiplihedron, the graph-associahedron, and numerous other associative objects have been proven to always form the face poset of some polytope, and determining whether the 2-associahedron is always a polytope has been the driving question of our work.

Another open question is the relationship between the 2-associahedron and other associative objects. We have seen that all multiplihedra are also constrainahedra, but we still do not know how the 2-associahedron fits into this picture. Are all constrainahedra also 2-associahedra? What about multiplihedra or graph-associahedra? If not, is there some bigger generalization of the associahedron which encompasses all of these objects?

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