It’s nice when a proof shows how to derive a result, instead of just simply proving the result. There is such a nice way to derive the divergence theorem in two and three dimensions. Here I describe a derivation for the two-dimensional case. The same type of proof works equally well for Green’s theorem itself, but the proof for an equivalent formulation known as the two-dimensional divergence theorem carries over directly to three-dimensions.

Recall the two formulations. Let \( S \) be a bounded region in the \( x-y \) plane, and let \( C = \partial S \), where \( C \) is a contour with the smoothness properties we’ve discussed in class. Then we have:

\[
\text{\textit{Green’s Theorem}}: \quad \oint_C \mathbf{F} \cdot \mathbf{t} \, ds = \iint_S \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dS \quad (1.1)
\]

and

\[
2-D \text{\textit{Divergence Theorem}}: \quad \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_S \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \, dS \quad (1.2)
\]

The integral on the left side of Eq. (1.1) gives the circulation of \( \mathbf{F} \) around \( C \), and the integral on the left side of Eq. (1.2) gives the flux of \( \mathbf{F} \) across \( C \).

We’ve spent a fair amount of time in class discussing surface integrals, so there should be nothing new or mysterious about calculating the right sides of the preceding two equations. But we haven’t said a lot about the line integrals on the left sides, so we’ll start with a brief discussion to recap what you may have seen before and what we have talked about in class in recent days.

\textbf{Short Tutorial on Line Integrals}

In a way, line integrals are a lot like surface integrals. We have worked with surface integrals like \( \iint_S f(x,y,z) \, dS \), where \( f(x,y,z) \) is a scalar function with domain that includes \( S \). To evaluate such an integral, we always fall back on a parameterization of the surface, say \( \mathbf{X}(s,t) = (x(s,t), y(s,t), z(s,t)) \). We then compute the cross product \( \mathbf{N} = \mathbf{X}_s \times \mathbf{X}_t \), estimate an element of surface area symbolized by \( dS = \|\mathbf{N}\|d\sigma \), and then write

\[
\iint_S f(x,y,z) \, dS \equiv \iint_D f(x(s,t), y(s,t), z(s,t)) \|\mathbf{N}\| \, d\sigma \quad (1.3)
\]

The parameterization \( \mathbf{X} \) makes it possible to evaluate the integral on the right side of Eq. (1.3). In fact, the integral on the right is the \textit{definition} of the intuitive expression on the left.
A similar set-up works for line integrals. Recall the concept of arc length. Imagine a curve $C$ in $\mathbb{R}^3$ and a parameterization of $C$

$$X(t) = (x(t), y(t), z(t)), t \in [a, b]$$

(1.4)

in which $X$ is continuously differentiable. The vector difference between two very close points $X(t_1)$ and $X(t_2)$ of $C$ can be estimated by

$$X(t_2) - X(t_1) \approx (\dot{x}(t_1), \dot{y}(t_1), \dot{z}(t_1)) \delta t$$

(1.5)

where $\delta t = t_2 - t_1$. Therefore, by the Pythagorean theorem, the length of arc on $C$ between $X(t_1)$ and $X(t_2)$ can be estimated by,

$$\delta s \approx \sqrt{\dot{x}(t_1)^2 + \dot{y}(t_1)^2 + \dot{z}(t_1)^2} \delta t = \|X(t)\| \delta t$$

(1.6)

On the one hand, the limit of the quotient $\delta s/\delta t$ yields the familiar expression for the derivative of arc length $\dot{s} = \sqrt{\dot{x}(t_1)^2 + \dot{y}(t_1)^2 + \dot{z}(t_1)^2}$, and on the other hand, Eq. (1.6) motivates the following definition for the path integral of the scalar field $f$ on $C$:

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \|\dot{X}(t)\| dt$$

(1.7)

Henceforth, we assume that $\|\dot{X}(t)\| > 0$ so that as $t$ increases, the point $\dot{X}(t)$ moves “forward” along $C$ without stopping or backing up. Let $t = \dot{X}(t)/\|\dot{X}(t)\|$ be the unit tangent vector pointing forward along $C$. This leads to the following important idea. Given a vector field

$$F(x, y, z) = P(x, y, z)i + Q(x, y, z)j + R(x, y, z)k$$

(1.8)

and a piecewise smooth closed curve $C$, we define the circulation of $F$ along $C$ as

$$\oint_C F \cdot t ds = \int_a^b F(x(t), y(t), z(t)) \cdot t \|\dot{X}(t)\| dt$$

$$= \int_a^b F(x(t), y(t), z(t)) \cdot \dot{X}(t) dt$$

(1.9)

If $C$ and the vector field $F$ lie in a plane, say in $\mathbb{R}^2$, we may also speak of the flux $\oint_C F \cdot n ds$ of a two-dimensional vector field across $C$, as in Eq. (1.2). This idea is developed in the next section.
Two-dimensional Divergence Theorem

Given the situation illustrated in the figure, we are going to calculate \( \oint_C \mathbf{F} \cdot \mathbf{n} \, ds \), the flux of \( \mathbf{F} \) across \( C \) in two ways. The figure shows a positively oriented curve \( C \) in \( \mathbb{R}^2 \) bounding a region denoted by \( D \). The positively oriented unit tangent vector to \( C \) is \( \mathbf{t} \), and the outward pointing unit normal is \( \mathbf{n} \). The figure illustrates a dissection of \( D \) into “small” rectangles and a few figures with rounded edges around the periphery of \( C \). Notice that each internal edge abuts on an adjacent edge of an adjacent figure. Suppose that the boundary of each figure is oriented positively. A few arrows have been drawn to show that in that case, the adjacent edges are oriented in opposite directions. This is true for all internal edges, so if we add the flux integrals for all the figures oriented in this way, the internal portions will cancel each other out, leaving just the flux integral over \( C \). That’s the first way we calculate the flux integral.

For the second way to calculate the flux integral, the idea is to imagine that, instead of just the crude dissection shown in the figure above, there are billions of tiny rectangles subdividing the interior of \( C \). When we estimate the flux integral of \( \mathbf{F} \) around the boundaries of one such rectangle, we can add up the results for all the rectangles to give us the two-dimensional divergence theorem.

Consider a particular rectangle of tiny dimensions \( \delta x \times \delta y \) centered at the point \((x, y)\). The approximate value of the flux outward across the each of the four edges beginning with the right edge and going counterclockwise is, respectively:

\[
\delta_x M \left( x + \frac{\delta_x}{2}, y \right), \quad \delta_y N \left( x, y + \frac{\delta_y}{2} \right), \quad -\delta_y M \left( x - \frac{\delta_y}{2}, y \right), \quad -\delta_x N \left( x, y - \frac{\delta_x}{2} \right) \tag{1.10}
\]
Regrouping and adding these estimates, we get an approximation for the net outflow from the rectangle:

\[ dF \approx \delta_x M \left( x + \frac{\delta_x}{2}, y \right) - \delta_x M \left( x - \frac{\delta_x}{2}, y \right) + \delta_y N \left( x, y + \frac{\delta_y}{2} \right) - \delta_y N \left( x, y - \frac{\delta_y}{2} \right) \]

\[ = \frac{\partial M}{\partial x} \delta_x \delta_y + \frac{\partial N}{\partial y} \delta_x \delta_y \]

Adding all such approximations for all the rectangles, and ignoring the negligible peripheral figures that lie adjacent to \( C \), we get a Riemann sum for a double integral

\[ \sum \left( \frac{\partial M}{\partial x} \delta_x \delta_y + \frac{\partial N}{\partial y} \delta_x \delta_y \right) \rightarrow \iint_D \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy \]  (1.12)

Then by equating Eq. (1.12) to the flux integral around \( C \), we arrive at the two-dimensional divergence theorem expressed by Eq. (1.2):

\[ \oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy \]  (1.13)

**What about Holes in the Domain?**

Now, let’s consider what happens when a region contains holes. Look again at the figure above and imagine that \( D \) has a hole in it. Take a pencil and draw such a figure, and denote the curve on the outer boundary of \( D \) by \( C_1 \), and denote the curve bounding the hole by \( C_2 \). Dissect \( D \) into rectangles oriented consistently with the positive orientation of \( C_1 \), and notice that when you draw orientation arrows on the edges of the figures that are adjacent to \( C_2 \), the latter curve inherits an orientation opposite to the sense of \( C_1 \).

**Gauss’ Divergence Theorem**

The argument for deriving the two-dimensional divergence theorem extends easily to the three-dimensional divergence theorem. Imagine that you have a solid \( V \) bounded by a surface \( S \) oriented so that the surface normal points outward away from \( V \), and suppose the domain of the smooth vector field \( \mathbf{F} \) includes \( V \), where

\[ \mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k} \]  (1.14)

Dissect \( V \) by rectangular parallelepipeds also oriented so that the normals to their surfaces point away from their interiors. Summing the flux of \( \mathbf{F} \) over all figures will lead to cancellations of the flux across the interior surfaces, leaving just the flux over \( S \). Moreover, an estimate of the outward flux from a typical parallelepiped can be made in
analogy with the way it was done for a rectangle in the two-dimensional case; if you work this out as an exercise, and you will see that you wind up with an approximation like
\[
\left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) \delta_x \delta_y \delta_z
\]
(1.15)

Summing over all the rectangular parallelepipeds, and ignoring the negligible fragmentary pieces that abut on S, you obtain a Riemann sum for a triple integral
\[
\sum \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) \delta_x \delta_y \delta_z \to \iiint_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dxdydz
\]
(1.16)
and finally arrive at Gauss’ divergence theorem:
\[
\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dxdydz
\]
(1.17)

What about a solid with holes in it? If the outer surface is oriented by an outward pointing normal, then the normal to the surface of the hole must point away from the interior of the solid and into the hole.

**Stokes’ Theorem**

I like to think of Stokes’ theorem as Green’s theorem on a potato chip. In other words, it’s just like Green’s theorem but instead of applying to just a flat bounded region in \( \mathbb{R}^2 \), it applies to a warped surface in \( \mathbb{R}^3 \). A derivation like above applies, but the details are more complicated. And instead of starting with the flux integral around the perimeter of the surface, we start with the circulation integral. I’d be glad to sketch it for you if you are interested, but you might prefer to derive the theorem yourself, for then you will see how the curl operator arises. By the way, notice that Stokes’ theorem for a bounded region of the \( x-y \) plane, applied to a vector field with a zero component in the \( z \) direction, is just Green’s theorem.