ODEs II, Lecture 1:  
**Homogeneous Linear Systems - I**

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**Introduction.** In the first lecture we discussed a system of linear ODEs for modeling the excretion of lead from the human body, saw how to transform a linear ODE of degree \( n \) into a linear system of \( n \) first-order ODEs, and noted the form of the general system of \( n \) first-order equations, for both linear and non-linear systems. We also solved a simple linear system of two first-order equations. Here we examine linear systems in greater detail.

**General Linear System.** Given a set of functions real-valued function \( a_{ij}(t) \) and \( g_i(t) \) defined on some interval \( I \) of the real line, where the indices \( i \) and \( j \) run from 1 to \( n \), we can write a system of first-order linear equations,

\[
\begin{align*}
\mathbf{x}(t) &= \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \\
A(t) &= \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \\
g(t) &= \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix},
\end{align*}
\]

\[
x'(t) = A\mathbf{x}(t) + g(t), \quad t \in I \subset \mathbb{R}
\]

The term \( g(t) \) is called the driving term or forcing term. If \( g(t) \equiv 0, t \in \mathbb{R} \) the system is called homogeneous, otherwise it is called non-homogeneous. In the former case, the system may be called a driven system, and in the latter case the system may be called undriven.\(^2\)

We begin our study with a homogeneous linear system with constant coefficients. When the value of the solution \( \mathbf{x}(t) \) is specified at an initial time \( t_0 \), the system is referred to as an initial value problem (IVP):

\[
\begin{align*}
\mathbf{x}(t) &= \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \\
A &= \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \\
\text{IVP:} \quad x' &= Ax, \quad \mathbf{x}(t_0) = \mathbf{x}^0
\end{align*}
\]

\(^1\) Based on a course given jointly with Michael Moody.

\(^2\) If both \( A \) and \( g \) are constant functions of \( t \), then the system is said to be autonomous. We will study nonlinear autonomous systems later in the course.
Note that, if the two vector functions $\mathbf{x}_1(t), \mathbf{x}_2(t)$ satisfy

$$\frac{d}{dt} \mathbf{x}_1(t) = \mathbf{x}_1'(t) = A\mathbf{x}_1(t), \quad \mathbf{x}_1(t_0) = \mathbf{x}_1^0$$

$$\frac{d}{dt} \mathbf{x}_2(t) = \mathbf{x}_2'(t) = A\mathbf{x}_2(t), \quad \mathbf{x}_2(t_0) = \mathbf{x}_2^0$$

then, defining

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$$

it follows that

$$\frac{d}{dt} \mathbf{x}(t) = \frac{d}{dt} (c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)) = c_1\mathbf{x}_1'(t) + c_2\mathbf{x}_2'(t)$$

$$= c_1 A\mathbf{x}_1(t) + c_2 A\mathbf{x}_2(t) = A\left(c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)\right) = A\mathbf{x}(t)$$

(1.5)

This is why the system is called \textit{linear}. Note that the initial conditions could also correspond: Then it can be seen by inspection that a solution to the IVP

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}^0 = c_1\mathbf{x}_1^0 + c_2\mathbf{x}_2^0$$

(1.6)

is obtained by summing the solutions of the two IVPs (1.3). This sum is in fact the \textit{unique} solution to IVP (1.6).\footnote{We will discuss the relevant Fundamental Existence and Uniqueness theorem in the next lecture.}

In case the matrix $A = A_{n \times n}$ has constant entries and $n$ linearly independent eigenvectors, there is an amazingly simple and direct way to solve the homogeneous system Eq. (1.2). Try $\mathbf{x}(t) = e^{rt}\mathbf{v}$ for some constant vector $\mathbf{v} = (v_1 \cdots v_n)^T$ and scalar $r$, in other words, let

$$\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = e^{rt} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

(1.7)

So

$$\mathbf{x}'(t) = re^{rt} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = re^{rt}\mathbf{v}$$

And,
Therefore, plugging these results into $x' = Ax$, we get:

$$e^{rt} Av = re^{rt}v$$

But $e^{rt} \neq 0$, therefore,

$$Av = rv$$

This shows that $v$ is an eigenvector of $A$ corresponding to the eigenvalue $r$, and, working backward with these values, we see that

$$x(t) = e^{rt}v = e^{rt}v \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} e^{rt}v_1 \\ \vdots \\ e^{rt}v_n \end{pmatrix}$$

is a solution of the system.

A remarkable feature of this derivation is that it converted a calculus problem into a linear algebra problem, in fact, an eigenvalue/eigenvector problem. This is great evidence for the usefulness of linear algebra, but there’s more to come.

For example, let’s see what happens when we have a complete set of eigenvectors for $A$, that is a set of eigenvectors that spans $\mathbb{R}^n$. To keep it simple, we’ll consider $\mathbb{R}^2$, but the approach is applicable to higher dimensions.

Example. Find linearly independent functions $x_1(t)$ and $x_2(t)$ that satisfy

$$\begin{align*}
 x_1' &= x_1 + x_2 \\
 x_2' &= 4x_1 + x_2
\end{align*}$$

We’ll solve this system in general then come back to the specific case:

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Rewrite Eqs. (1.12) in vector/matrix notation,

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
Our approach is to find eigenvalues and corresponding eigenvectors of the matrix $A$. In order to do so, we first find the eigenvalues of $A$:

\[
\begin{vmatrix}
1 - \lambda & 1 \\
4 & 1 - \lambda
\end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0
\]

\[
\Rightarrow (\lambda_1, \lambda_2) = (3, -1)
\]

Corresponding eigenvectors $v$, where $A v = \lambda v$, can be found using a familiar row-reduction procedure symbolized as follows:

\[
\begin{align*}
\lambda_1 = 3: & \\
\begin{bmatrix}
1 - 3 & 1 \\
4 & 1 - 3
\end{bmatrix} & \rightarrow \begin{bmatrix}
-2 & 1 \\
4 & -2
\end{bmatrix} & \rightarrow \begin{bmatrix}
-2 & 1 \\
0 & 0
\end{bmatrix} & \Rightarrow v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
\lambda_2 = -1: & \\
\begin{bmatrix}
1 + 1 & 1 \\
4 & 1 + 1
\end{bmatrix} & \rightarrow \begin{bmatrix}
2 & 1 \\
4 & 2
\end{bmatrix} & \rightarrow \begin{bmatrix}
2 & 1 \\
0 & 0
\end{bmatrix} & \Rightarrow v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}
\end{align*}
\]

It is obvious on inspection that the eigenvectors at the right-hand side of Eq. (1.16) are linearly independent, but even without looking we know they must be because they belong to distinct eigenvalues.\(^4\) Now, in accord with our construction, $x(t) = e^t v$, where now we see that $v$ and $r = \lambda$, as found in Eq. (1.16). So, let

\[
x_i = \begin{pmatrix} x_{i1} \\ x_{i2} \end{pmatrix} = e^{\lambda_i} v = e^{3t} \begin{pmatrix} 1 \\ 2 \\ 2e^{3t} \end{pmatrix}
\]

In the representation $x_{ij}$ above, the second index indicates the vector identification number and the first index gives the component number for that vector. For example, below when referencing the vector $x_2$, we will use $x_{21}$ and $x_{22}$ to indicate the components of $x_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$. The reason for choosing this order of the index will become clear below when we place both vectors in a common matrix, so that the second index will indicate the column in which the vector appears. For economy of expression, we often omit explicit statement of the dependency of variables like $x$ and $y$ on the independent variable $t$.

We have explained why Eq. (1.17) must be a solution of Eq. (1.12); but since ODE work is often error prone, it’s a good idea to doublecheck the result by direct calculation. From Eq. (1.17),

\[
x_i' = \begin{pmatrix} x_{i1}' \\ x_{i2}' \end{pmatrix} = \begin{pmatrix} 3e^{3t} \\ 6e^{3t} \end{pmatrix}
\]

\(^4\) Note that we could pick any scalar multiples of the eigenvectors shown; to simplify subsequent arithmetic, we picked multiples to ensure that all components of the vectors are integers.
Therefore,
\[
x_{11}' = 3e^{3t} = e^{3t} + 2e^{3t} = x_{11} + x_{21}
\]
\[
x_{21}' = 6e^{3t} = 4e^{3t} + 2e^{3t} = 4x_{11} + x_{21}
\]
(1.19)

Thus we confirm that \( \mathbf{x}_1 = \begin{pmatrix} e^{3t} & 2e^{3t} \end{pmatrix}^T \) solves the system, as we knew it should.

Now let
\[
\mathbf{x}_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = e^{3t} \mathbf{v}_2 = e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}
\]
(1.20)

We could easily confirm that \( \mathbf{x}_2 \) is also a solution of Eq. (1.12), so now we have two distinct solutions of our homogeneous equation:
\[
\mathbf{x}_1(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}
\]
(1.21)

Neither of these solutions satisfies the initial conditions of Eq. (1.13). But they are linearly independent functions of \( t \) for \( t \in \mathbb{R} \), as is easily verified in this case by noting simply that \( \mathbf{x}_1 \) cannot be a fixed linear multiple of \( \mathbf{x}_2 \) valid for all \( t \in \mathbb{R} \).\(^5\) Because the solutions are linearly independent, they can be combined to produce a general solution of Eq. (1.12), as demonstrated next.

As noted above, any linear combination of solutions of a homogeneous linear system of ODEs must also be a solution. Therefore, any linear combination of the solutions in Eq. (1.21) must satisfy the homogeneous Eq. (1.14). So, all we have to do is see whether we can match the given initial conditions to some linear combination of \( \mathbf{x}_1(t_0), \mathbf{x}_2(t_0) \). This translates into the question of whether we can find constants \( c_1 \) and \( c_2 \) such that
\[
c_1 \mathbf{x}_1(t_0) + c_2 \mathbf{x}_2(t_0) = c_1 e^{3t_0} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t_0} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_1 e^{3t_0} \\ c_2 e^{-t_0} \end{pmatrix} = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}
\]
(1.22)

Generally, any \( n \)-by-\( n \) matrix comprised of \( n \) columns of eigenvectors belonging to distinct eigenvalues must be non-singular. So, the two-by-two matrix in Eq. (1.22) consisting of two such eigenvectors must be non-singular. In our present case, this is obvious, but it will usually not be obvious in higher dimensions, so it is good to keep the general condition in mind. In any case, the linear independence of the columns in the

\(^5\) Later we will discuss a Wronskian test, similar to the one encountered in Math 13 but tailored for linear systems, which will permit us to determine linear dependence or independence of a set of \( n \) solutions of a homogeneous linear system of \( n \) equations.
square matrix of Eq. (1.22) implies that the equation has a unique solution for $c_1 e^{3t_0}$ and $c_2 e^{-\beta t_0}$, hence a unique solution for $c_1$ and $c_2$ because $e^{3t_0} \neq 0$, $e^{-\beta t_0} \neq 0$. In other words, Eq. (1.22) can be solved for arbitrary initial conditions. This proves that

$$x(t) = c_1 x_1(t) + c_2 x_2(t)$$

(1.23)

is a general solution of the system of Equation (1.14). The solution we have found is also complete, i.e., there aren’t any other solutions than the ones we have found. This follows from a Fundamental Existence and Uniqueness Theorem for systems of ODEs alluded to in a previous footnote.

Let’s proceed to the particular case anticipated above. Using the initial values $x(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ in Eq. (1.22) we obtain

$$\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{pmatrix} c_1 e^{3t_0} \\ c_2 e^{-\beta t_0} \end{pmatrix} = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

(1.24)

Therefore, the solution of the IVP with initial conditions defined by Eq. (1.13) is

$$x(t) = 2e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{-\beta t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2e^{3t} + e^{-\beta t} \\ 4e^{3t} - 2e^{-\beta t} \end{pmatrix}$$

(1.25)

and

$$x(t) \approx 2e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ as } t \rightarrow \infty$$

(1.26)

We see in this example the important role that linear algebra can play in the study of differential equations.

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6 Notice that we have introduced some discriminating vocabulary here. A general solution of an ODE or system of ODEs is one that provides a solution for any arbitrary set of initial conditions. A complete solution is one that provides all possible solutions. The distinction arises because it is possible in some circumstances to have more than one solution for a given initial value problem, so that a general solution (providing only one solution for any given set of initial values) isn’t necessarily complete. For the problems encountered in this course, the general solution will also be the complete solution.