Homogeneous Linear Systems - II

Mike Raugh

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**Introduction.** We discuss some fundamental theory of systems of linear ODEs without proofs, including the use of the *Wronskian* to determine whether a set of solutions of a linear system is a *fundamental set*, i.e., a basis for the solution space of the system of equations. The Wronskian to appear here is similar to the Wronskian for a single homogeneous linear equation introduced in Math 13, but here the same idea is adapted to a homogeneous system of linear equations.

Then we apply the theory with examples of systems in which the coefficient matrix has: (1) distinct eigenvalues, (2) eigenvalues of multiplicity greater than 1 (non-defective case), and (3) complex eigenvalues and eigenvectors. Items 2 and 3 anticipate homework that will appear in the second assignment. The more challenging case, in which the coefficient matrix has a defective eigenspace, will also appear later.

Further discussion of the theory, with some proofs, will be given in the next set of notes.

**Setting the Stage.** We will deal with equations of a form introduced in the previous lecture:

\[ x' = Ax + g \]  \hspace{1cm} (2.1)

where

\[ x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} a_{11}(t) & a_{1n}(t) \\ \vdots & \ddots \\ a_{n1}(t) & a_{nn}(t) \end{pmatrix}, \quad g(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix} \]  \hspace{1cm} (2.2)

and it is assumed that the entries of the matrices \( A \) and \( g \) are continuous functions on some interval \( I = (a, b) \subset \mathbb{R} \). The term *homogeneous equation* then refers to the case for which \( g \) is identically 0 at all points of the interval \( I \).

We take for granted a fundamental existence and uniqueness theorem and a corollary to be specified after the following reminder. We conclude with three characteristic examples.

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1 Based on a course taught jointly with Michael Moody.
2 Recall that a *defective* eigenspace is one whose dimension is smaller than the multiplicity of its eigenvalue.
Linear Dependence/Independence. A set of vectors \( \{v_1, \ldots, v_n\} \subset V \), where \( V \) is a vector space, is said to be \textit{linearly dependent} if there exist corresponding scalars \((c_1, \ldots, c_n)\) not all equal to zero such that

\[
c_1v_1 + \cdots + c_nv_n = 0
\]  

(2.3)

The point I want to emphasize is that the right-hand expression of Eq. (2.3) represents the "zero" vector of \( V \): \( \mathbf{0} \in V \). It is a special vector in \( V \). Do not confuse it with the scalar zero ("0"). If the only set of scalars \((c_1, \ldots, c_n)\) that satisfies Eq. (2.3) is all zeroes, then the set \( \{v_1, \ldots, v_n\} \) is said to be \textit{linearly independent}.\(^3\)

Why make a point of such a familiar definition? In complicated cases such as we will consider in this lecture, it is important to keep clearly in mind the distinction between the scalar zero and the null vector ("zero") on the right-hand side of Eq. (2.3). Let’s see why this matters.

We will want to test whether a given set of solutions of a system of ODEs such as Eq. (2.2) is linearly independent. Then any one of the solutions will consist of \( n \) functions, regarded as entries of a column vector. Assuming that the \( j \)th solution is \( x_j \), we can have

\[
x_j = \begin{pmatrix} x_{j1}(t) \\ \vdots \\ x_{jn}(t) \end{pmatrix}
\]  

(2.4)

Since \( x_j \) assigns for each value of \( t \in I \) a point in \( \mathbb{R}^n \), it is a function from the domain \( I \) to the co-domain \( \mathbb{R}^n \). This kind of function is referred to as a \textit{vector-valued} function.

Now, if we say that we have any set of \( m \) linearly dependent vector-valued functions \( \{x_1, \ldots, x_m\} \) like (2.4) defined on an interval \( I \), we mean that some non-trivial set of scalars \((c_1, \ldots, c_n) \neq (0, \ldots, 0)\) satisfies

\[
c_1x_1(t) + \cdots + c_mx_m(t) = 0, \quad t \in I
\]  

(2.5)

Otherwise, if only the trivial set of scalars, i.e., \((c_1, \ldots, c_n) = (0, \ldots, 0)\), can satisfy Eq. (2.5), we say that the set \( \{x_1, \ldots, x_m\} \) is \textit{linearly independent}. In this case the vector \( \mathbf{0} \) on the right-hand side of Eq. (2.5) is the \( n \)-dimensional vector-valued function defined by

\[^3\text{Note that we abuse set notation in this lecture. When we denote a set of scalars as } \{c_1, \ldots, c_n\} \text{ we intend that the scalars be taken in the order specified, and we allow for repetition of a given value. This contravenes proper set-theoretic usage in which order and repetition make no difference.}\]
This makes it clear that linear dependence for vector-valued functions like (2.4) places a very powerful constraint on the set of scalars \( \{c_1, \ldots, c_n\} \), since it means that Eq. (2.5) must be satisfied at all points of the interval \( I \) with a fixed set of scalars. Contrast this with the situation in which the same set of vectors \( \{\mathbf{x}_1, \ldots, \mathbf{x}_m\} \) is linearly dependent at a single point \( t_0 \in I \). This is a far less demanding condition, since the scalars are constrained at only one point:

\[
c_i \mathbf{x}_i(t_0) + \cdots + c_m \mathbf{x}_m(t_0) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}
\]  
(2.7)

Even the requirement that a set of vector-valued functions \( \{\mathbf{x}_1, \ldots, \mathbf{x}_m\} \) be linearly dependent at every single point of an interval \( I \) does not guarantee that the set is linearly dependent over the entire interval. The following two vector-valued functions afford a decisive illustration:

\[
\mathbf{z}_1 = \begin{pmatrix} 1 \\ t \end{pmatrix}, \quad \mathbf{z}_2 = \begin{pmatrix} t \\ t^2 \end{pmatrix}
\]  
(2.8)

At any fixed point \( t_0 \in \mathbb{R} \), the two vectors \( \{\mathbf{z}_1(t_0), \mathbf{z}_2(t_0)\} \) are obviously linearly dependent – the one is just \( t_0 \) times the other. But the only two fixed scalars \( \{c_1, c_2\} \subset \mathbb{R} \) such that the following equality holds for all \( t \),

\[
c_1 \begin{pmatrix} 1 \\ t \end{pmatrix} + c_2 \begin{pmatrix} t \\ t^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}
\]  
(2.9)

must be \( c_1 = c_2 = 0 \), as you can see, for example, by considering the case \( t = 0 \).

Linear dependence of a set of vector-valued functions on an interval is a far stronger constraint than linear dependence at each separate point of the interval. However, it is a striking and extremely useful fact that if the vector-valued functions are solutions of some homogeneous system of linear ODEs on some interval \( I \), then linear dependence of the set at just one point of \( I \) is sufficient to ensure linear dependence on the entire interval. This will be explained below.
**Fundamental Theory.** Suppose in the following two theorems, which we state without proof, that the matrix $A$ and vector-valued function $g$ are as in Eq. (2.2), where the entries of $A$ and $g$ are all continuous functions on the interval $I = (a, b) \subset \mathbb{R}$.

Theorem 1: Fundamental Existence and Uniqueness. The IVP corresponding to Eq. (2.2) in which we specify $t_0 \in I = (a, b)$ and

$$x(t_0) = x^0 = \begin{pmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{pmatrix}, \quad x^0 \in \mathbb{R}^n$$

has a unique solution

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \forall t \in (a, b)$$

Theorem 2. The set of solutions of the homogeneous system of Eqs. (2.2), i.e., $g = 0$, is a vector space of dimension $n$. Any set of $n$ linearly independent solutions $\{x_1(t), \ldots, x_n(t)\}$ is a basis, and such a set is called a fundamental set or basic set for the homogeneous system.

Comment 1. We accept Theorem 1 without proof, and we will prove Theorem 2 as a corollary of Theorem 1 in the notes accompanying Lecture 3.

Comment 2. We will see in the notes accompanying Lecture 3 that a fundamental set of solutions can be used to find a general solution of the homogeneous IVP. Later we will also see how a fundamental solution lends itself to solving the non-homogeneous equation using the method of variation of parameters.

**The Wronskian for Homogeneous Linear Systems.** In a vector space it is useful to be able to determine whether a given set of vectors is a basis for the space. The Wronskian serves that purpose for solutions of linear ODEs. The Wronskian for a set of vector-valued functions, defined next, can be used to determine whether a set of solutions is a fundamental set. It is similar to the Wronskian for solutions of an $n^{th}$ order linear equation introduced in Math 13, and it serves a similar purpose. We state the following facts about this new Wronskian and supply proofs in the next set of notes.

**Definition.** The Wronskian of $n$ $n$-dimensional vectors is their determinant. In other words, suppose $\{x_1(t), \ldots, x_n(t)\}$ are $n$-dimensional vectors. Then
Theorem 3 (Wronskian for Systems). Let \( S = \{x_1(t), \ldots, x_n(t)\} \) be a set of solutions of the homogeneous system (2.2), then \( S \) is linearly dependent iff the Wronskian \( W[x_1, \ldots, x_n](t) \) vanishes for some \( t_0 \in I \). In that case, the Wronskian vanishes for all \( t \in I \). Conversely, the set \( S \) is linearly independent if the Wronskian is non-zero for some \( t_0 \in I \), in which case the Wronskian does not vanish for any \( t \in I \).

Comment. The preceding theorem can be proved directly from Theorem 1, and it would be a good test of your understanding to work out such a proof. (A proof is given in the notes accompanying Lecture 3.) However, see the comment that follows the next theorem.

We have the following generalization of Abel’s theorem

Theorem 4 (Liouville’s generalization of Abel’s theorem, constant coefficient case). Suppose the \( n \)-dimensional vector functions \( \{x_1(t), \ldots, x_n(t)\} \) are each solutions of the homogeneous linear equation with constant coefficient matrix \( A \):

\[
x' = Ax, \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad a_{ij} \in \mathbb{R}
\] (2.13)

Then

\[
W[x_1, \ldots, x_n](t) = ce^{\text{Tr}(Ax)} = ce^{\left(\sum_{j=1}^{n} a_{jj}\right) t}
\] (2.14)

Comment. Note that Theorem 4 implies Theorem 3 for the constant-coefficient case. The next theorem, the industrial-strength version of Liouville’s theorem, implies Theorem 3 in full generality.

Theorem 5 (Liouville’s generalization of Abel’s Theorem). Let \( \{x_1(t), \ldots, x_n(t)\} \) be any \( n \) solutions of the homogeneous Eq.(2.2), then

\[
W[x_1, \ldots, x_n](t) \equiv W(t_0)e^{\int_{t_0}^{t}(\text{Tr}Ax)dt} = W(t_0)e^{\int_{a}^{b}(\sum_{j=1}^{n} a_{jj})dt}, \quad \{t_0, t\} \subseteq (a, b)
\] (2.15)

\[\text{See your text for a reminder of Abel’s Theorem.}\]
Comment. It is an immediate consequence that \( \{x_i(t), \ldots, x_n(t)\} \) is a fundamental set iff 
\( W[x_i, \ldots, x_n](t) \neq 0, \forall t \in (a, b) \). Moreover, once zero always zero, i.e., if \( W \) vanishes for any \( t \in (a, b) \) then it vanishes for all \( t \) in the interval.

Examples. Recall the method we used to solve the linear system with constant coefficient matrix \( A \), as in Eq. (2.13)

\[
\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad a_{ij} \in \mathbb{R} \quad (2.16)
\]

We tried
\[
\mathbf{x} = e^{rt} \mathbf{v} \quad (2.17)
\]
where \( r \) and \( \mathbf{v} \) are constants: \( r \in \mathbb{R} \) and \( \mathbf{v} \in \mathbb{R}^n \). We found that for \( r \) an eigenvalue of \( A \), and \( \mathbf{v} \) any eigenvector corresponding to \( r \), Eq. (2.17) will yield a solution to Eq. (2.16).

The problem is that we need to find a fundamental set, i.e, \( n \) linearly independent solutions of \( A \), so the technique for using Eq. (2.17) may break down if \( A \) does not have \( n \) linearly independent eigenvectors. Let’s look at some examples to help make this clear.

Example 1.

\[
\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \quad (2.18)
\]

Find eigenvalues and eigenvectors for \( A \). Recall that the trace of a matrix is the sum of the diagonal entries. The characteristic equation of the given matrix is:

\[
r^2 - (\text{Tr} \, A) r + (\text{det} \, A) = r^2 - (-5) r + 4 = (r+1)(r+4) = 0 \Rightarrow \{r_1, r_2\} = \{-1, -4\} \quad (2.19)
\]

So by row echelon reduction,

\[
r_1 = -1: \begin{pmatrix} -3+1 & \sqrt{2} & 0 \\ \sqrt{2} & -2+1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathbf{v}_1 \propto \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \quad (2.20)
\]

where the symbol “\( \propto \)” means “proportional to.” Similarly
\[ r_2 = -4: \quad \mathbf{v}_2 \propto \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} \] (2.21)

Thus we have the two solutions
\[ \mathbf{x}_1 = e^{-t} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \quad \mathbf{x}_2 = e^{-4t} \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} \] (2.22)

We can see in this simple two-dimensional case that the one solution is not a constant multiple of the other, hence we may conclude easily that the two solutions are linearly independent. This direct method doesn’t work in higher dimensions, but the Wronskian method always works. So we demonstrate it in these examples. By the Wronskian theorem demonstrated above, we can check the value of the Wronskian of the two functions (2.22) at any (convenient) point to determine dependence/independence:

\[
W[\mathbf{x}_1, \mathbf{x}_2](0) = \begin{vmatrix} e^{-t} & -e^{-4t} \sqrt{2} \\ e^{-t} \sqrt{2} & e^{-4t} \end{vmatrix} = \begin{vmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{vmatrix} = 1 \neq 0 \] (2.23)

So \( \{\mathbf{x}_1, \mathbf{x}_2\} \) is a fundamental set, and therefore linear combinations provide the general solution:
\[ \mathbf{x} = c_1 e^{-t} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} \] (2.24)

Example 2: This next example illustrates the case of a multiple root (eigenvalue) for which the corresponding eigenspace is non-defective.

\[ \mathbf{x}' = A \mathbf{x}, \quad A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \] (2.25)

\( A \) is symmetric, so recall from Math 63 that the spectral theorem for real symmetric matrices guarantees that we have all real eigenvalues and a complete set of real eigenvectors. Let’s check it out. The characteristic equation is
\[ (r - 2)(r + 1)^2 = 0 \] (2.26)

\[ \Rightarrow \{r_1, r_2\} = \{2, -1\} \]

Using row echelon reduction (omitting details), we find
We deal with the multiple case more explicitly:

\[
\begin{bmatrix}
0 -(-1) & 1 & 1 & 0 \\
1 & 0 -(-1) & 1 & 0 \\
1 & 1 & 0 -(-1) & 0
\end{bmatrix}
\]

\[ r_2 = -1 : \]

\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \Rightarrow v_1 + v_2 + v_3 = 0
\]

So, the eigenspace for \( r_2 = -1 \) is a plane through the origin. We can choose any two linearly independent vectors in the plane as representative eigenvectors, for example:

\[
v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}
\]

These results yield three solutions for Eq. (2.25):

\[
x_1 = e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad x_2 = e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad x_3 = e^{-t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}
\]

We check whether the Wronskian of these solutions vanishes at a convenient point, namely \( t = 0 \):

\[
W[x_1, x_2, x_3](0) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix} = -1 \neq 0
\]

It doesn’t vanish, therefore, this verifies that \( \{x_1, x_2, x_3\} \) is a fundamental set. We can now state that a general solution of the homogeneous system is

\[
x = c_1e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3e^{-t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}
\]
Example 3: The third and last example illustrates the case in which the eigenvalues and eigenvectors are complex. The method is similar to the preceding examples, but the arithmetic is complicated by the complex values that enter the derivation.

General Observations. The gist of the following comments is contained in Eqs. (2.37) and (2.38). We suppose that $A$ is real, and, to keep it simple, we also suppose that $A = A_{2 \times 2}$. \(^5\)

$$x' = Ax$$ (2.33)

and that $r_i = \lambda + i\mu$, $\mu \neq 0$ is an eigenvalue of $A$ and $v_i = a + ib \in \mathbb{C}^2$ is a corresponding eigenvector. Note then that

$$Av_i = r_i v_i \quad \Rightarrow \quad \overline{Av_i} = A\overline{v_i} = \overline{r_i} v_i$$ (2.34)

So $r_2 = \lambda - i\mu$ is also an eigenvalue of $A$, and $v_2 = a - ib$ is a corresponding eigenvector. And, because we assumed that $\mu \neq 0$ ensuring that the two eigenvalues are distinct, the set of eigenvectors $\{v_1, v_2\}$ must be linearly independent. So we may conclude that

$$W[x_1, x_2](0) = \det[(a + ib) (a - ib)] \neq 0$$, and therefore the two solutions

$$x_1 = e^{(\lambda + i\mu)t} (a + ib), \quad x_2 = e^{(\lambda - i\mu)t} (a - ib)$$ (2.35)

are linearly independent.

Now each of $x_1$ and $x_2$ is the other’s conjugate:

$$\overline{x_1} = e^{(\lambda + i\mu)t} (a + ib) = [e^{it} (\cos \lambda t + i \sin \lambda t)] (a - ib)$$
$$= [e^{it} (\cos \lambda t - i \sin \lambda t)] (a - ib) = e^{(\lambda - i\mu)t} (a - ib) = x_2$$ (2.36)

so half their sum must be a real solution equal to just the real part of $x_1$, and similarly, half their difference must be an imaginary solution equal to just the imaginary part. This tells us that we can get two solutions directly by breaking $x_1$ into its real and imaginary parts

$$x_1 = e^{(\lambda + i\mu)t} (a + ib)$$
$$= e^{it} [a \cos \mu t - b \sin \mu t] + ie^{it} [a \sin \mu t + b \cos \mu t]$$ (2.37)

yielding the set of two real solutions $\{u_1(t), u_2(t)\}$ of Eq. (2.33):

\(^5\) With minor modification, the discussion generalizes to more general real coefficient matrices $A = A_{m \times n}$. 

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We know that \( \{u_1(t), u_2(t)\} \) are linearly independent because their span must equal the span of \( \{x_1, x_2\} \), and the latter are linearly independent as shown above.

We note in passing the interesting fact that the vectors \( \mathbf{a} \) and \( \mathbf{b} \) in Eq. (2.35) must be linearly independent. This is because, as stated above, the set of vectors \( \{\mathbf{a} + i\mathbf{b}, \mathbf{a} - i\mathbf{b}\} \) is linearly independent, and

\[
\text{Span} \{\mathbf{a}, \mathbf{b}\} = \text{Span} \{\mathbf{a} + i\mathbf{b}, \mathbf{a} - i\mathbf{b}\}
\]  

(2.39)

This concludes our general observations for the complex case.

**Complex Application.** Let’s use Eqs. (2.37) and/or (2.38) to solve a problem.

Suppose

\[
x' = Ax, \quad A = \begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix}
\]  

(2.40)

The characteristic equation of \( A \) is

\[
r^2 - (\text{Tr } A) r + (\det A) = r^2 + r + \frac{5}{4} = 0
\]

\[
\Rightarrow \quad r_1 = -\frac{1}{2} + i, \quad r_2 = -\frac{1}{2} - i
\]  

(2.41)

By the general reasoning above, we only need to work with one of the eigenvalues, say \( r_1 \):

\[
\begin{pmatrix}
-1/2 - (-1/2 + i) & 1 & 0 \\
-1 & -1/2 - (-1/2 + i) & 0
\end{pmatrix} \rightarrow
\begin{pmatrix}
-i & 1 & 0 \\
-1 & -i & 0
\end{pmatrix} \Rightarrow \quad \mathbf{v}_1 \propto \begin{pmatrix} 1 \\ i \end{pmatrix}
\]

(2.42)

Now split \( \mathbf{v}_1 \) into real and imaginary parts to get

\[
\mathbf{x}_1 = e^{\frac{-1}{2} i t} \begin{pmatrix} 1 \\ 0 + i \end{pmatrix}
\]  

(2.43)
Then expand the right-hand Eq. (2.43) into its real and imaginary parts \(\{u_1, u_2\}\), which themselves are guaranteed to be a pair of linearly independent solutions. Or, since the general formula is close at hand, we can compare Eq. (2.43) with Eq. (2.37) to find

\[
\lambda = -\frac{1}{2}, \mu = 1; \quad a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

(2.44)

Plugging these values into Eq. (2.38), we get the same set of solutions:

\[
\begin{align*}
\mathbf{u}_1 &= e^{\frac{t}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t = \begin{pmatrix} e^{\frac{t}{2}} \cos t \\ -e^{\frac{t}{2}} \sin t \end{pmatrix} \\
\mathbf{u}_2 &= e^{\frac{t}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t = \begin{pmatrix} e^{\frac{t}{2}} \sin t \\ e^{\frac{t}{2}} \cos t \end{pmatrix}
\end{align*}
\]

(2.45)

We can apply the Wronskian to confirm a fact we already deduced in generality above, namely, that \(\{u_1, u_2\}\) is a linearly independent set:

\[
W[u_1, u_2](0) = \begin{vmatrix} e^{\frac{t}{2}} \cos t & e^{\frac{t}{2}} \sin t \\ -e^{\frac{t}{2}} \sin t & e^{\frac{t}{2}} \cos t \end{vmatrix}_{t=0} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0
\]

(2.46)

In each of the examples above, the coefficient matrix was constant, real and (importantly) had a complete set of eigenvectors. Unfortunately, not all linear systems are so simple. In a later lecture, we will deal with the case of coefficient matrices that have a defective eigenspace. The existence of such matrices is one of the features that make linear algebra so interesting and challenging.