Introduction. In a recent lecture we discussed phase portraits, and I distributed the Borrelli-Coleman description of a phase portrait gallery for two-dimensional linear autonomous systems with a constant coefficient matrix, which covered all cases except the case of a zero eigenvalue. I also discussed the case of a zero eigenvector.

We have talked about a fundamental set of solutions for a homogeneous system of linear ODEs, i.e., any basis for the solution space of the system. I underscored the very important fact that the Wronskian of a fundamental set can never vanish. Moreover, the Wronskian is an indicator of whether a set of solutions is a fundamental set, depending on whether the Wronskian of the set of solutions does or does not vanish.

Today we develop the idea of a fundamental matrix for a given linear system of ODEs, which is a matrix comprised of an arbitrary fundamental set of solutions for the system. We will see that a fundamental matrix affords a convenient, consolidated way to think about IVPs, and we will encounter a special fundamental matrix that satisfies a particularly simple initial value problem. This note concludes with a reprise of the fact that a coefficient matrix with a full set of eigenvectors leads to a decoupled system of ODEs, which we examined previously from the point of view of a trial solution \( x = e^t v \).

Fundamental Matrix

Recall that a fundamental set of solutions for a homogeneous system of linear ODEs

\[
\begin{array}{c}
\mathbf{x}' = A\mathbf{x}, \\
A = \begin{pmatrix} a_{11}(t) & a_{1n}(t) \\ \vdots & \ddots \\ a_{n1}(t) & a_{nn}(t) \end{pmatrix}
\end{array}
\] (6.1)

is a set of \( n \) linearly independent solutions of the system of ODEs, say \( \{x_1, \ldots, x_n\} \). If all the entries of \( A \) are continuous on an interval \( I = (\alpha, \beta) \subset \mathbb{R} \), a Fundamental Existence and Uniqueness Theorem for systems of homogeneous linear systems of ODEs tells us that the set of solutions of Eq. (6.1) is a vector space of vector-valued functions defined on \( I \) of dimension \( n \). In that case, we know that \( \{x_1, \ldots, x_n\} \) is a basis for the solution space. So the general solution on \( I \) for ODE (6.1) is

\[
\mathbf{x} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n, \quad \{c_j\} \text{ are scalars}
\] (6.2)
The general solution can be condensed further in matrix notation by defining

\[
\begin{bmatrix}
\begin{array}{c}
\mathbf{c} \\
\vdots \\
\mathbf{c}_n
\end{array}
\end{bmatrix}, \quad \Psi(t) \equiv \begin{bmatrix} x_{1,1}(t) & \cdots & x_{1,n}(t) \\
\vdots & \ddots & \vdots \\
x_{n,1}(t) & \cdots & x_{n,n}(t)
\end{bmatrix}
\]

So we can represent the general solution of Eq. (6.1) compactly as

\[\mathbf{x}(t) = \Psi(t)\mathbf{c}\]  \hspace{1cm} (6.3)

where \(\mathbf{c}\) represents a vector of arbitrary constants.

Now consider the IVP

\[\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}^0 = \begin{bmatrix} x_1(t_0) \\
\vdots \\
x_n(t_0) \end{bmatrix}\]  \hspace{1cm} (6.4)

We know that IVP (6.4) has a unique solution, and therefore there is some vector, say \(\mathbf{c}_0\), such that the solution given by Eq. (6.3) is

\[\mathbf{x}(t) = \Psi(t)\mathbf{c}_0\]  \hspace{1cm} (6.5)

Recall that the Wronskian of a fundamental set can never vanish. In the present case the columns comprising \(\Psi(t)\) are a fundamental set, \(W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = \det \Psi(t)\), and, therefore, \(\Psi(t)\) is non-singular for all \(t\) – in particular it is non-singular for \(t = t_0\). In the instance of IVP (6.4), this observation allows us to solve Eq. (6.5) for \(\mathbf{c}_0\) in terms of \(\Psi(t)\):

\[\mathbf{x}(t_0) = \Psi(t_0)\mathbf{c}_0 = \mathbf{x}^0 = \begin{bmatrix} x_1(t_0) \\
\vdots \\
x_n(t_0) \end{bmatrix}\]  \hspace{1cm} (6.6)

yielding

\[\mathbf{c}_0 = \Psi^{-1}(t_0)\mathbf{x}_0 = \Psi^{-1}(t_0)\begin{bmatrix} x_1(t_0) \\
\vdots \\
x_n(t_0) \end{bmatrix}\]  \hspace{1cm} (6.7)

Putting Eqs. (6.3) and (6.7) together, we have
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\[ x(t) = \Psi(t)c_0 = \Psi(t)\Psi^{-1}(t_0)x^0 \]
\[ = \Psi(t)\Psi^{-1}(t_0)\begin{bmatrix} x_1(t_0) \\ \vdots \\ x_n(t_0) \end{bmatrix} \]  \hspace{1cm} (6.8)

This gives the unique general solution of Eq. (6.4) for arbitrary initial conditions in a condensed form. But the formulation can be consolidated even more.

**A Special Fundamental Matrix**

The preceding development is valid for any fundamental matrix \( \Psi(t) \) for Eq. (6.1). But there are infinitely many fundamental matrices. In fact, if \( B = B_{nn} \) is an arbitrary matrix, then \( \Psi_1(t) = \Psi(t)B \) is a fundamental matrix iff \( B \) is non-singular. This is because

\[
\det \Psi_1(t) = (\det \Psi(t))(\det B) \tag{6.9}
\]

Therefore, if the right-hand side is non-zero, the same must be true of the left-hand side; and if \( \det \Psi_1(t) \neq 0 \), then \( \det B \neq 0 \) because by hypothesis \( \det \Psi(t) \neq 0 \).

Given the abundance of fundamental matrices for a given system of equations, it is a striking fact that, no matter which one you start with, say \( \Psi(t) \), a unique fundamental matrix can be defined by \(^1\)

\[
\Phi(t) = \Psi(t)\left[\Psi(0)\right]^{-1} = \left[x_1(t) \ldots x_n(t)\right]\left[x_1(0) \ldots x_n(0)\right]^{-1} \tag{6.10}
\]

Here’s why \( \Phi(t) \) is well-defined and unique:

- \( \Phi(t) \) is well-defined because, as mentioned above, the Wronskian of a fundamental matrix can never vanish, so \( \Psi(0) = \left[x_1(0) \ldots x_n(0)\right] \) is non-singular and invertible.

- The columns of \( \Phi(t) \) are solutions of (6.1) because each column, being a linear combination of the solutions \( \{x_1(t) \ldots x_n(t)\} \), is itself a solution.

\(^1\) Note that your text’s definition is more general: \( \Phi(t) = \Psi(t)\left[\Psi(t_0)\right]^{-1} \) for a given \( t_0 \in I \). This could (and perhaps should) be made more explicit by writing \( \Phi(t_0,t) = \Psi(t)\left[\Psi(t_0)\right]^{-1} \) for the general case, reserving \( \Phi(t) \) for the case \( t_0=0 \) as we do here.
• $\Phi(t)$ is a fundamental matrix because its Wronskian is nonzero:

$$W[\Phi](0) = \det \left( \begin{bmatrix} x_1(0) & \cdots & x_n(0) \end{bmatrix} \begin{bmatrix} x_1(0) & \cdots & x_n(0) \end{bmatrix}^{-1} \right)$$

$$= \det I_n = 1$$  
(6.11)

• $\Phi(t)$ is unique because it is evident from Eq. (6.10) that $\Phi(0) = I_n$. Therefore, we have:

$$\Phi' = A\Phi, \quad \Phi(0) = I_n$$  
(6.12)

Thus, the $j$-th column of $\Phi(t)$ is the unique solution of the IVP

$$x' = Ax, \quad x(0) = e_j$$  
(6.13)

where $e_j$ is the $j$-th standard basis vector in $\mathbb{R}^n$, so $\Phi(t)$ is uniquely defined by Eq. (6.13).\(^2\)

We mention the important special case in which the coefficient matrix $A$ is constant. In that case the matrix $\Phi(t)$ has special properties for which $\Phi$ is called the matrix exponential of $A$, as will be discussed in the next lecture.

**Example.** Let’s set up the complete solution for the following IVP using the special fundamental matrix $\Phi$,

$$x' = Ax, \quad A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

(6.14)

In this case, the coefficient matrix is constant, so, according to the definition above, we can refer to $\Phi$ as the “matrix exponential of $A$.” The characteristic equation for $A$ is

$$r^2 + r - 6 = (r + 3)(r - 2) = 0$$

$$(r_1, r_2) = (-3, 2)$$  
(6.15)

Corresponding eigenvectors can be found

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\(^2\) Recall that the standard basis vectors are those in which the only non-zero entry is 1. They are arranged so that the $j$-th one has 1 as its $j$-th entry.
which we can use to form a fundamental matrix

$$
\Psi(t) = \begin{pmatrix}
1 & e^{3t} \\
-4 & e^{2t}
\end{pmatrix}
$$

(6.17)

Therefore,

$$
\Phi(t) = \Psi(t)\Psi^{-1}(0) = \begin{pmatrix}
e^{-3t} & e^{2t} \\
-4e^{-3t} & e^{2t}
\end{pmatrix}^{-1}
$$

$$
= \begin{pmatrix}
e^{-3t}/5 + 4e^{2t}/5 & -e^{-3t}/5 + e^{2t}/5 \\
-4e^{-3t}/5 + 4e^{2t}/5 & 4e^{-3t}/5 + e^{2t}/5
\end{pmatrix}
$$

(6.18)

Because $\Phi(t)$ is unique, you would wind up with the same matrix $\Phi(t)$ no matter what fundamental matrix $\Psi(t)$ for $A$ you started out with. To conclude the example, note that

$$
\Psi(t)\Psi^{-1}(t_0) = \left[ \Phi(t)\Psi(0) \right] \left[ \Psi^{-1}(0)\Phi^{-1}(t_0) \right] = \Phi(t)\Phi^{-1}(t_0)
$$

(6.19)

so,

$$
x(t) = \Psi(t)\Psi^{-1}(t_0) \begin{pmatrix}
x_1(t_0) \\
\vdots \\
x_n(t_0)
\end{pmatrix} = \Phi(t)\Phi^{-1}(t_0) \begin{pmatrix}
x_1(t_0) \\
\vdots \\
x_n(t_0)
\end{pmatrix}
$$

(6.20)

In the next lecture, we will define the matrix exponential $e^{At}$ for constant coefficient matrices and show that it is equal to $\Phi(t)$. Then we will see that the key expression in Eq. (6.20) simplifies to $\Phi(t)\Phi^{-1}(t_0) = \Phi(t-t_0)$, eliminating the need to invert a matrix with non-constant entries. More about that later….

Matrix Diagonalization and Decoupled Linear Systems

Consider

$$
x' = Ax, \quad A = A_{n \times n}, \quad A = \text{constant}
$$

(6.21)

where $A$ has $n$ linearly independent eigenvectors, say $\{v_1, \ldots, v_n\}$ with corresponding eigenvalues $\{r_1, \ldots, r_n\}$. Previously we have shown that a fundamental matrix for Eq. (6.21) can be constructed as follows
But we will circumvent that result and solve the system of Eq. (6.21) in another way that reveals another aspect of the structure of the system of equations, namely, that the system can be decoupled in a natural way because the coefficient matrix can be diagonalized. So let

\[ S = [v_1 \ \cdots \ v_n] \]  

(6.23)

where \( S \) is nonsingular, and \( y(t) \) can be defined by

\[ x(t) = S y(t) \]

\[ \Leftrightarrow \]

\[ y(t) = S^{-1} x(t) \]

(6.24)

Differentiating the first Eq. (6.24) gives us

\[ (S y)' = S y' = x' = Ax = AS y \]

\[ \Rightarrow \]

\[ S y' = AS y \]

\[ \Rightarrow y' = S^{-1} AS y \]

(6.25)

But

\[ S^{-1} AS = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_n \end{bmatrix} = D \]  

(6.26)

So, if

\[ y = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix} \]

(6.27)

then Eq. (6.25) yields the decoupled system of scalar equations

\[ y'_1 = r_1 y_1 \]

\[ \vdots \]

\[ y'_n = r_n y_n \]

(6.28)

each of which can be solved separately. Such a system is called decoupled because the solution of each equation has been decoupled from the solution of any other equation of the system. We are familiar with the solutions:
Therefore, recalling the definition of $y$ from Eq. (6.24), we get at last,

$$y = \begin{pmatrix} c_1 e^{\rho t} \\ \\
\vdots \\ \\
c_n e^{\rho t} \end{pmatrix}$$

We have seen this solution before, where it was noted that any IVP associated with Eq. (6.30) can be solved uniquely because the Wronskian $W[v_1, \ldots, v_n](t)$ is non-zero for all values of $t$. For example, to test you understanding, show how to specify a unique set of constants \{c_1, \ldots, c_n\} that satisfies Eq. (6.21) with the initial conditions

$$x(t_0) = \left( x_1^0, \ldots, x_n^0 \right)^T,$$

where \( (t_0, x_1^0, \ldots, x_n^0) \in \mathbb{R}^{n+1} \) is an arbitrary set of real numbers.

A Passing Comment

In previous lectures we have discussed the fact that not all matrices are diagonalizable, so it is not possible to decouple an arbitrary system in the manner shown above. However, it is always possible to convert any linear system with constant coefficients into linear cascades. A cascade of the type relevant to our ODEs is a system in which each ODE of the system involves only two dependent variables, namely, the one that appears differentiated on the left and the one that appears differentiated in the preceding equation.\footnote{The order of precedence may go from first to last, as expressed here, or last to first, depending on the way in which the system is laid out.} The first equation is an exception because it involves only the first dependent variable. A cascade can be solved by working in reverse order. The reason it is possible to break an arbitrary linear system into cascades rests upon the Jordan normal form of a matrix, which is the key to determining a complete set of generalized eigenvectors for a linear system of ODEs. This matter is sketched in a supplemental note placed online with this lecture for your interest only, to give you a glimpse of what is really involved in the deeper study of generalized eigenvectors. You are not required to read the supplemental note.