Advice for Drawing Phase Portraits

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April 3, 2004

Here's a way to organize your sketching of a phase portrait. The steps below are suggested just to help you get started. But after some you have had some experience, you will be able to draw phase portraits for the ones shown in the Borrelli-Coleman portrait gallery without going through all of the steps.

I assume the constant $2 \times 2$ coefficient matrix of your ODE system is non-singular and that zero is not an eigenvalue. For the latter case, see your text or work it out yourself.

I find it all too easy to make a messy portrait while figuring out which direction the orbits are flowing and that it helps to use colored pencils to distinguish the various structures of an initial draft. I suggest using nullclines to help determine flow directions, but ordinarily these would not appear in your final figure unless specifically called for in an exercise.

You can use the following suggestions to make a first draft. Afterwards you can easily sketch a good phase portrait without the clutter of all the extra paraphernalia.

1) Use (up to) 5 differently colored pencils for: axes [lead]; critical point(s) [whatever1]; eigenline(s), if any [whatever2]; one each for the nullclines [whatever3 and whatever4].

2) Draw axes.

3) Circle critical point(s).

4) Draw eigenline(s), if any, label each with the associated e-value, and put arrows on them indicating flow direction(s).

5) Draw both nullclines, and color them differently. (There are two nullclines if the coefficient matrix is nonsingular.) Note that a nullcline divides the plane into two sectors such that the flow direction for the coordinate corresponding to the eigenline is opposite on either side of the line. For example, the $x_1$-nullcline divides the plane into one half on which the $x_1$-coordinate of an orbit increases, and the other half on which the $x_1$-coordinate of an orbit decreases.

6) In each of the four sectors bounded by the nullclines, and for each coordinate variable, draw an arrow indicating the direction of flow for the corresponding coordinate. Place the two arrows butt-to-butt, and draw attention to them by drawing two circles around them colored with the two colored pencils used for the nullclines. This little symbol shows you in what general direction an orbit must move in that sector, for example, "up and to the left." Notice
that once you have figured out the general flow direction for a sector, the flow direction in
the opposite sector is exactly opposite ("down and to the right")!

7) If your critical point is a node, figure out or memorize the angles of attack an orbit can make
when emanating from, or flowing towards, a neighborhood of the origin. (See the example at
the end.) Remember, a proper (star) node, (improper) node and deficient node have their
characteristic behaviors near the origin and on their way to infinity - you need to indicate
these on your portrait. Otherwise, be able to recognize a center or spiral point.

8) Use the preceding information to draw some characteristic non-linear orbits, placing arrows
on them to indicate the direction of flow.

9) Be sure to write somewhere on your portrait what type of node you have.

10) If your portrait is neat and nicely indicative of the characteristic flow determined by your
equation, congratulate yourself. These problems are error prone and require careful attention
and a cool hand.

Example. As an example, here’s a way to work out the slope of attack for an orbit of a deficient
node. See your text and/or Borrelli-Coleman for a different kind of explanation. Suppose the
general solution is of the form

\[
\begin{pmatrix}
    x_1(t) \\
    x_2(t)
\end{pmatrix} = x(t) = e^{\lambda t} \left[ c_1 \alpha + c_2 \left( \beta + \alpha \right) \right] = e^{\lambda t} \left[ c_1 \left( \frac{\alpha_1}{\alpha_2} \right) + c_2 \left( \frac{\beta_1}{\beta_2} \right) + c_2 t \left( \frac{\alpha_1}{\alpha_2} \right) \right]
\]

(1.1)

So \( x_1 = e^{\lambda t} L_1 \) and \( x_2 = e^{\lambda t} L_2 \), where \( L_1 \) and \( L_2 \) are linear functions of \( t \) discernable by inspection
of Eq. (1.1), and \( \dot{x}_1 = e^{\lambda t} \left( \lambda L_1 + \dot{L}_1 \right) \) and \( \dot{x}_2 = e^{\lambda t} \left( \lambda L_2 + \dot{L}_2 \right) \). Therefore,

\[
\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{\lambda L_2 + \dot{L}_2}{\lambda L_1 + \dot{L}_1}
\]

Because \( \dot{L}_1 \) and \( \dot{L}_2 \) are constants, it is clear now that only the coefficients of \( t \) in \( L_1 \) and \( L_2 \) will
matter in the limit as \( t \to \infty \), so

\[
\lim_{t \to \infty} \frac{dx_2}{dx_1} = \lim_{t \to \infty} \frac{L_2}{L_1} = \frac{\alpha_2}{\alpha_1}
\]

(1.2)

Therefore the angle of attack of an orbit at the origin (and asymptotically at infinity) is tangent to
(in the same direction as) the eigenline, and vertical in case the denominator of Eq. (1.2)
vanishes. Similar derivations can be applied for proper and improper nodes.