ODEs II, Lecture 12:

The Nonlinear Pendulum – Ideal Frictionless Point Mass Case

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**Introduction.** We discuss an ideal pendulum as a transitional case. Since we know the physics, we can speculate how the phase portrait looks. But we can also do the math based on the fact that we can integrate and analyze the equation for the direction field of the phase portrait. We’ll look at the problem both ways. But on Friday we’ll see a general method for solving problems for which the latter advantage may be absent, using *linearization*.

**Nonlinear Pendulum.** We idealize a pendulum as a point-mass $m$ suspended at the tip of a rigid weightless pendulum arm of length $L$, free to rotate without friction in a vertical plane. We model this system with a left-hand Cartesian reference coordinate system in which the positive $y$-axis points down vertically and the $x$-axis points horizontally to the right. The angle $\theta$ in radians measured counterclockwise gives the angle of the pendulum from the vertical $y$-axis. Regarding $\theta = \theta(t)$ as a function of time, we compute the position of the point-mass at time $t$ as:

$$
\mathbf{x} = \begin{pmatrix} L \sin \theta \\ L \cos \theta \end{pmatrix} \quad (12.1)
$$

Applying Newton’s law $ma = F$ to the tangential component of the motion at the tip of the pendulum, since that is the cause of acceleration, we have (using the dot product)

$$
mL\dot{\theta} = \begin{pmatrix} 0 \\ mg \end{pmatrix} \cdot \begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix} \Rightarrow \ddot{\theta} + \frac{g}{L} \sin \theta = 0 \quad (12.2)
$$

We note in passing that for small $\theta$ we obtain an approximate linear equation, which can be solved in elementary terms:

$$
\ddot{\theta} + \frac{g}{L} \theta = 0 \quad (12.3)
$$

In Friday’s lecture we will see a systematic way to linearize equations near a critical point.

Going back to Eq. (12.2), which cannot be solved in elementary terms, we convert it to a two-dimensional system of first-order ODEs in normal form. Letting $u = \theta$ and $v = u'$, we get
\begin{align*}
u' &= v \\
v' &= -\frac{g}{L} \sin u \tag{12.4}
\end{align*}

Let’s try to infer a phase portrait for this system. Time has been removed from the phase plane, so things will look different than what you may imagine for the pendulum swinging in time and space. It might help in what follows to make sketches alongside of the explanation.

As usual, we start with the nullclines. The only \( u \)-nullcline in the phase space for this system is given by the equation \( u = 0 \), i.e., the \( v \)-axis. And the only \( v \)-nullclines are where \( u = k\pi, k \in \mathbb{N} \). Since the points \((k\pi,0)\) exhaust all intersections of a \( u \)-nullcline with a \( v \)-nullcline, they comprise the equilibrium (i.e., critical or stationary points). The even values of \( k \), i.e., \( u = 2k\pi \), correspond to the positions in which the pendulum hangs straight down. And the odd values, \( u = (2k+1)\pi \), correspond to “straight up.”

**Heuristic Approach.** We begin by relying on our sense of how a real pendulum works. Considering the pendulum modeled by the system of Eqs. (12.4), it’s reasonable to assume that the equilibria at even multiples of \( \pi \) are stable, and orbits near a stable equilibrium are periodic, implying they are simply-connected closed curves. Moreover, the flow directions within sectors bounded by the nullclines suggest that the closed orbits flow clockwise. So you can envision closed orbits around every even multiple of \( \pi \). But there is a limited region around each equilibrium in which there can be closed orbits, and the next two paragraphs explain why.

Think about the unstable equilibria at odd multiples of \( \pi \). Here we imagine a *magic* orbit corresponding to the ideal case in which the pendulum falls slowly counterclockwise away from its upright position at such a rate that it never quite gets back to the upright position again, and it converges on the upright position as time goes to infinity. Such an orbit can exist only in our minds, but it’s reasonable to speculate that the mathematical model allows it. Notice that such an orbit, when run backwards in time, also converges on the upright position, but in the reverse direction. So you can have a magic orbit arising from an unstable equilibrium, arcing upwards and above the next stable equilibrium to the right, then descending and converging on (but not reaching) the next unstable equilibrium to the right. And, similarly, beginning at the unstable equilibrium to the right, there’s a magic orbit that corresponds to the pendulum moving in the reverse direction (i.e., clockwise). This orbit moves to the left, dropping downward initially until it is precisely beneath the equilibrium mentioned before, then it turns upward and converges on the original unstable equilibrium. Now you have a convex closed curve comprised of two unstable equilibria and magic flows connecting them, one above the \( u \)-axis and one below. In other words, this convex closed curve is made up of four orbits (counting equilibria). And they bound all the closed (periodic) orbits inferred in the preceding paragraph. There is one of these closed curves between any two consecutive unstable equilibria, and they all look exactly alike. You can now imagine...
that these curves go towards (and away from) an unstable equilibrium, possibly like the four arms of a saddle point. We’ll confirm this latter assumption below.

Finally, we know from considering the physical pendulum, there must be orbits that correspond to a pendulum moving so rapidly that it keeps on spinning, either clockwise or counterclockwise. Since the flow direction of an orbit can never reverse itself, an orbit of a perpetually spinning (not oscillating) pendulum can never cross the \( u \)-axis. Such orbits, when above the \( u \)-axis must flow rightward above the \( u \)-axis or leftward below the \( u \)-axis. Flows like these are periodic in the \( u \)-direction, like waves that rise and fall in correspondence with the magic closed curves beneath (or above) them.

A sketch of the description above should make it clear what the phase portrait of System (12.4) should look like. But now let’s check it out mathematically.

**Analytic Approach.** Because the system (12.4) is autonomous, we can compute the direction field:

\[
\frac{dv}{du} = - \frac{g \sin u}{L} v
\]  

Moreover, Eq. (12.5) is separable and easily integrated:

\[
\frac{1}{2} v^2 - \frac{g}{L} \cos u = c
\]  

where \( c \) is a constant of integration. By recognizing that \( \int \sin u \, du \) is proportional to potential energy, we can convert Eq. (12.6) into the law of conservation of energy. Just add \( \alpha^2 = g/L \) to both sides to obtain

\[
E(u, v) \equiv \frac{1}{2} v^2 + \alpha^2 \int_0^u \sin s \, ds = c_0
\]  

The orbits therefore lie on level curves of \( E(u, v) \).

The question we need to answer is whether the orbits consist of entire level curves or just portions of level curves. Referring to Eq. (12.6), we see that the equation

\[
v^2 = \frac{2g}{L} \cos u + c = \frac{2g}{L} (\cos u + \beta)
\]  

can plausibly represent points in phase space only for \( \beta \geq -1 \), since for \( \beta < -1 \) the expression on the right-hand side is negative for all \( u \) and therefore no \( v \in \mathbb{R} \) can satisfy
the equation. In fact, because the values of $\cos u$ are restricted to the interval $[-1,1]$, we can analyze System (12.4) in terms of the values of $\beta$ arranged in four distinct cases:

1. If $\beta = -1$, then Equation (12.8) is satisfied at all the “even-numbered” equilibrium points (i.e., at $u = 2k\pi$, $k \in \mathbb{N}$), and nowhere else.

2. If $-1 < \beta < 1$, then Eq. (12.8) has two distinct solutions $u_1 > 0$, $u_2 = -u_1$ strictly between $-\pi$ and $\pi$. Neither of the points $(u_1,0)$ nor $(u_2,0)$ is an equilibrium. And, for all values $u \in [u_1,u_2]$, the right-hand side of Eq. (12.8) yields real-valued solutions $\pm v(u)$. Since no such solution point can be an equilibrium point, the level curve corresponding to the given value of $c$ must be a closed orbit that is symmetric about the $u$-axis. Note that the orbit circulates about the origin. Moreover, translates of this orbit left and right by multiples of $2\pi$ are also orbits of the system (12.4). By noting the flow directions in the sectors defined by the nullclines described above, it can be seen that the flow around each orbit is clockwise.

3. If $\beta = 1$, then the locus of points satisfying Eq. (12.8) is again a closed curve around the origin, as in the previous case. But in this case the two points closest to the origin at which $v = 0$ are equilibrium points: $(-\pi,0)$ and $(\pi,0)$. So here we have the situation in which a closed curve is not just a single orbit. The other two orbits on this curve are the arcs flowing from one equilibrium to the other. Translates of this curve left and right by multiples of $2\pi$ yield additional sets of orbits of system (12.3). Note that the orbits discussed in cases 1-3 cover the $u$-axis completely. This fact makes the next case obvious, but we will argue it again in the same vein as the others.

4. If $\beta > 1$, then the right-hand side of Eq. (12.8) can never be zero, so the locus of solutions never crosses the $u$-axis. These orbits are wavy periodic curves that extend to both ends of the $u$-axis. Each orbit comes in a matching pair; the one above the axis flows from left to right, and the one below flows from right to left. These results confirm the results arrived at above heuristically. But we can go farther. We can deduce the tangent of the angle of approach a “magic” orbit makes as it approaches an unstable equilibrium. We want to compute the limit of $dv/du$ as $u \to (2k+1)\pi$. There are four magic orbits abutting on a given unstable equilibrium, so we have to distinguish among them. Assembling results from Eqs. (12.5) and (12.8), we find

$$\frac{dv}{du} = -\frac{L}{2g} \left( \frac{\sin u}{\pm \sqrt{\cos u + 1}} \right), \quad u \neq (2k+1)\pi \quad (12.9)$$

where we take the plus sign in the denominator for a curve above the $u$-axis and the negative sign for a curve below. As an example, we’ll compute the slope of the upper curve as $u$ approaches $\pi$, distinguishing the approaches from the left and right. Because $\sin u$ is positive as $u$ approaches $\pi$ from the left and negative as $u$ approaches $\pi$ from
the right, care must be taken to distinguish the limit of the expression on the right in the following expression:

$$\lim_{u \to \pi} \frac{dy}{du} = -\frac{L}{2g} \lim_{u \to \pi} \frac{\sin u}{\sqrt{\cos u + 1}}$$  \hspace{1cm} (12.10)

Using the first non-zero terms of Taylor’s expansion for the sine and cosine at $u = \pi$, we find that,

$$\lim_{u \to \pi} \frac{\sin u}{\sqrt{\cos u + 1}} = \lim_{u \to \pi} \frac{\pi - u}{|\pi - u|/\sqrt{2}}$$  \hspace{1cm} (12.11)

The last limit takes opposite signs, depending on the direction of approach of $u$ to $\pi$, with the result that

$$\lim_{u \to \pi^-} \frac{dy}{du} = -\frac{L}{g}, \quad \lim_{u \to \pi^+} \frac{dy}{du} = \frac{L}{g}$$  \hspace{1cm} (12.12)

Similarly, to compute the comparable limits for the lower magic orbit, we take the negative sign in the denominator of Eq. (12.9) and obtain results with opposite signs,

$$\lim_{u \to \pi^-} \frac{dy}{du} = \frac{L}{g}, \quad \lim_{u \to \pi^+} \frac{dy}{du} = -\frac{L}{g}$$  \hspace{1cm} (12.13)

The calculations demonstrate that the magic curves approach (and recede from) an unstable equilibrium analogous to the way that the two eigenlines cross at a saddle point of a constant-coefficient system of two ODEs.

**Conclusion.** The example of the ideal pendulum is advantageous for two reasons. We can sketch a phase portrait based on our knowledge of how real pendulums behave; from that understanding we can speculate confidently about what an ideal pendulum should do. And, as it turns out, a fairly complete mathematical analysis is possible because the direction-field equation is separable and can be solved, yielding orbital constraints that we can analyze in detail. These two advantages are not always available, and therefore it is useful to have more general analytical tools to assist in developing a phase portrait. In the next lecture, we will talk about *linearization* as such a tool. We will see that linearization utilizes Taylor’s expansion again, but in a more systematic way than was used going from Eq. (12.10) to Eq. (12.11).

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1. Taylor’s expansion of a function $f$ at $a \in \mathbb{R}$ is $f(x) = f(a) + f'(a)(x-a)/! + f''(a)(x-a)^2/2! + \cdots$. If the series converges, then in some neighborhood of $a$ the error in the approximation using the initial polynomial of degree $n$ is of order $(x-a)^{n+1}$. We know from calculus that Taylor’s expansion for $\sin x$ and $\cos x$ converge for any $\{a, x\} \in \mathbb{R}$, e.g., $a=\pi$. 