Some Symbols and Nomenclature Used in Lectures

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In order to make it easier to state facts about mathematics, I use the following symbols and nomenclature. My usage isn’t always consistent and formal, as it would have to be in a text on mathematical logic or axiomatic set theory. It’s just a matter of speeding up communication. This note documents some of this language of convenience.

• I use \( \mathbb{R}^m \) to denote \( m \)-dimensional real space, by which I mean the vector space of all real \( m \)-tuples, which can be regarded as \( m \)-dimensional real vectors. I often assume that this space is Euclidean \( n \)-space \( E^n \), which means that the length of a vector is given by the “Euclidean metric,” i.e. the Pythagorean theorem.

• As a convention, I usually think of the vectors of \( \mathbb{R}^m \) as column vectors.

• Individual vectors may be represented by a lower case letter either bolded or with an “arrow” on top: \( \mathbf{x} \) or \( \vec{x} \) or even by an unadorned italic \( x \) if the intent is clear from context.

• In a vector space of dimension \( k \), I will sometimes write \( \mathbf{0} \) or \( \vec{0} \) as the zero (null) vector. In order to emphasize that we refer to the null vector in \( \mathbb{R}^k \), I may include a subscript: \( \mathbf{0}_k \) or \( \vec{0}_k \).

• The symbol “\( \propto \)” reads “is proportional to.” For example, if for two vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) we write \( \mathbf{v}_1 \propto \mathbf{v}_2 \), then it is understood that there exist nonzero scalar \( \alpha \) such that

\[
\mathbf{v}_1 = \alpha \mathbf{v}_2
\]

• The symbol “\( \equiv \)” is used to connect a new term (on the left-hand side) to its definition on the right-hand side. For example, we may write:

\[
\text{Nul}(A) \equiv \text{NS}(A) \equiv \text{Null Space of matrix } A, \text{ where } A \text{ is, say, } m\text{-by-}n. \text{ Using set notation explained below, the null space of } A \text{ is defined by}
\]

\[
\text{Nul}(A_{m,n}) \equiv \left\{ \mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}_n \right\}
\]

• For emphasis, I will sometimes specifically reference the dimensions of a matrix by subscripting it or its symbol, as for example I may symbolize an \( m \)-by-\( n \) matrix \( A \) by...
writing $A_{m,n}$. If the subscript is omitted, and it usually will be, it is because I believe the context makes it clear what the dimensions are.

- I often write algebraic expressions involving matrices and scalars in which I use no special convention to indicate what is a one-dimensional column vector or a higher-dimensional matrix, or to distinguish a scalar from a vector. In such cases, I am assuming it is clear from context what the terms are and that the matrices are compatible for the given operations. But usually, capital letters will represent matrices, and unadorned lower-case letters will represent scalars. For example,

$$AX = B$$
$$A\vec{x} = c\vec{b}$$
$$A\mathbf{x} = c\mathbf{b}$$

- An expression like $c_1\mathbf{x} + c_2\mathbf{y}$ represents a linear combination of vectors $\mathbf{x}$ and $\mathbf{y}$.

- I often abbreviate “linearly independent” as “LI” or “L.I.” And “linear dependent” as “LD” or “L.D.”

- Logic. I sometimes use logical notation:

$$\therefore, \exists, \exists!, \forall, \neg, \land, \lor$$

for “therefore,” “there exists,” “there exists a unique,” “for all,” “not,” “and,” and “or,” respectively.

I sometimes abbreviate “such that” as “st” or “s. t.”

- Sets of objects (“elements”) are sometimes denoted by enclosing symbols for the objects between braces. For example $\{c_1,\ldots,c_k\}$ refers to the set containing objects denoted by $c_1,\ldots,c_k$, whatever they may be. The set so designated does not depend on the order in which the elements are arranged, and elements can be repeated without changing the set: two sets are identical iff they contain the same elements.

- An ordered set is a set of elements arranged in a specific order. An ordered set can be denoted by exhibiting the terms in order between regular parentheses: for example, $(a,b,\ldots,w)$ for a finite set, or $(x_1,x_2,\ldots)$ for an infinite sequence of elements. But notice that, if an integer subscript is included, it is assumed that the index increases in the same order as the elements of the set, so we could write the latter set as $\{x_1,x_2,\ldots\}$ without loss of clarity.
• The symbol $\in$ is used to denote membership in a set, e.g., $c_i \in \{c_1, \ldots, c_k\}$. We use the symbol $\notin$ to indicate that a thing does not belong to a set. For example, we might have a situation in which $a \notin \{c_1, \ldots, c_k\}$. Set membership is a primitive concept: It’s not defined in terms of anything else.

• Set inclusion is denoted by “$\subset$”, so for two sets of elements $P$ and $Q$ we could have $P \subset Q$. This just means that every element of $P$ is also an element of $Q$, or, expressed differently, $(\forall x)[(x \in P) \Rightarrow (x \in Q)]$. In such a case, we say that “$P$ is a subset of $Q$.”

• A word about logical implication. The logical implication “$A \Rightarrow B$” is defined to mean “$B \lor \neg A$”, i.e., “Either $B$ is true or $A$ is false.” I stress this because of the following item.

• The definition of set inclusion leads to a tricky point, interesting perhaps only to mathematicians. The empty set $\emptyset$ by definition is the set for which, no matter what $x$ is, the statement “$x \in \emptyset$” is FALSE. Thus, according to the definition of implication given in the previous item, $(x \in \emptyset) \Rightarrow (x \in Q)$ is trivially TRUE. So for any set $Q$, $\emptyset \subset Q$. In other words, $\emptyset$ is a subset of (is included in) every set. However, here’s the subtlety: $\emptyset$ may or may not belong to (i.e., be an element of) a given set. For example, $\emptyset \subset \mathbb{R}$ but $\emptyset \notin \mathbb{R}$. On the other hand, for the set $\{\emptyset\}$ it is true that $\emptyset \in \{\emptyset\}$. This is a case where you have to pay careful attention to definitions. The expressions “included in” means something different from “belongs to” or “is an element of.”

• A figure of speech arises from the situation described in the previous item. If the set inclusion $P \subset Q$ is TRUE because $P = \emptyset$, we say that the inclusion is “vacuously satisfied.” As an example, we could say “truly” that all the extra-terrestrials in attendance at Harvey Mudd College have won the Putnam Exam at least once. In other words, the statement “the set of extra-terrestrials at HMC is a subset of the set of Putnam winners” is formally TRUE as a matter of logic. By a generalization of this idea, we might say that the implication $p \Rightarrow q$ is vacuously satisfied, meaning that $p$ is FALSE. This oddity of language is basic and convenient as a logical device, but it takes getting used to.

• Set membership is often indicated by the “curly brackets” notation in either of two ways: by extension, in which the individual elements are enumerated, as in $\{a, b, d, 5, M\}$, where the letters $a, b, d$, and $M$ refer to specific items and 5 refers to the actual number, or by intention, i.e., specifying the property that determines whether an element belongs to the set, as in the following, which specifies the set of even integers.
\{m : (\exists n \in \mathbb{N})[m = 2n]\}

or

\{m \mid (\exists n \in \mathbb{N})[m = 2n]\}

- So now we can say that elements, say \(x\) and \(y\), belong to a set, say \(A\), in various ways. Two examples are

\((x \in A) \land (y \in A)\)

and

\(\{x, y\} \subseteq A\)

For a more specific example, I may indicate that a certain set of symbols, such as \(c_1, \ldots, c_k\), represent real scalars by writing \(\{c_1, \ldots, c_k\} \subseteq \mathbb{R}\) or complex scalars by writing \(\{c_1, \ldots, c_k\} \subseteq \mathbb{C}\). This symbolism means that each of the elements referenced in the set \(\{c_1, \ldots, c_k\}\) is a real or complex number, respectively.

- Continuous real-valued functions on an interval, say \((a, b)\) or \([a, b]\) can be represented by \(C(a, b)\) or \(C[a, b]\), respectively. Real-valued functions on the real line with continuous first derivative, or continuous first and second derivatives, can be represented by \(C^{(1)}(-\infty, \infty)\) or \(C^{(2)}(\mathbb{R})\), etc.

- **Cardinality.** Two sets are said to have the same cardinality if there is a one-to-one mapping of the one onto the other. If the sets are finite, then saying they have the same cardinality is equivalent to saying they have the same number of elements. As a convention, for arbitrary sets we may say of two sets that have the same cardinality that they have the same number of elements.

Some common abbreviations are:

- ODE, Ordinary Differential Equation
- IVP, Initial Value Problem
- BVP, Boundary Value Problem
- PDE, Partial Differential Equation
- FEUT, Fundamental Existence and Uniqueness Theorem