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Cake-Cutting Algorithms: Be Fair If You Can. By Jack Robertson and William Webb. A K Peters, 1998, x + 181 pp., $38.

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> *Many feel that mathematics will receive as much of its direction and vigor in the future from problems in the social sciences as it has received in the past from the physical sciences.* (p. 2)

As I grow older, I find myself yearning more and more for a connection between my work and the complex problems facing the world. And unlike most people, I am in a profession where I have the flexibility to choose what I do. Should this yearning influence the kind of research that I pursue? Should it affect how and what I teach my students? At the very least I should vigilantly ponder these questions.

Because of this yearning, I have taken up the challenge of a problem motivated by the social sciences—the problem of *fair division.* Mathematically, it is a very rich subject with many applications and a variety of open questions, ranging from very deep problems to accessible ones that my undergraduates can pursue. A very nice survey of the subject can be found in *Cake-Cutting Algorithms*, by Jack Robertson and William Webb.

The basic question—*how to cut a cake fairly?*—is surely an ancient one. However, Robertson and Webb's book demonstrates that the subject has evolved considerably in the fifty years since Steinhaus [10] first posed the question as a serious mathematical endeavor. Now much more than a collection of ad hoc results, the subject has matured, exhibited fertile connections with many areas of mathematics, and proved itself practical in social applications [4], [8].

The question is loaded with terms that must be made precise. In today's theory, "cake" could mean any desirable set of goods (or burdens, or mixtures of goods and burdens), each with various *properties* (such as being divisible or indivisible) and *restrictions* (such as the number of goods a player may get). The word "cut" refers to kinds of divisions that can be carried out in practice, such as discrete procedures or continuous moving-knife schemes for real cakes, and compensation procedures for the division of estates (using money as a divisible medium to negotiate the allocation of indivisible objects). The question “how?” suggests that we are not only interested in the existence of solutions, but desire *constructive* procedures for finding them. And there are many possible “fairness” notions. Two of the better known examples are *proportional* (or *simple fair*) division, in which each of n players feels she received at least $1/n$ of the cake, and the stronger notion of *envy-free* division, in which each person feels that no one else received a larger piece.

The main issues in the field are the following:

1. (Existence) Does a division with the desired fairness property *exist*?

2. (Construction) Is there a discrete or continuous *procedure* for locating solutions? If not, can one locate an approximate solution?

3. (Properties) Can one give bounds for the number of steps or cuts required? What can be accomplished with a limited number of cuts? What kinds of procedures are provably impossible?

4. (Implementation) Which procedures are useful in applications such as divorce settlements and estate division? Are they convincingly fair to each player? Do they induce players to be truthful? Can negotiators, lawyers, lay people, or even countries use them to resolve conflicts?

As a mathematical survey, *Cake-Cutting Algorithms* focuses primarily on issues (1), (2), and (3). The extensive mathematical development of (3) and the emphasis on cake division distinguish it from an earlier book [3] that takes a complementary and wider view of the subject by additionally addressing (4) while scaling back discussion of mathematical issues.

Ideas and techniques for solving fair division questions come from many different arenas. The following are some of my favorite examples.

Measure Theory Let C denote the cake to be divided. A natural model for a player's valuation of the cake is a probability measure on subsets of C. Thus $\mu_i(A)$ represents the value of subset A to player i as a fraction of the whole cake. The cake-cutting problem becomes non-trivial and interesting only when the player valuations differ; however, one lesson of the subject, noted by Steinhaus and developed further by Robertson and Webb, is the “serendipity of disagreement”. Players can often get away with more if there are differences in player opinions.

As an example, consider the following nice result of Barbanel [1]. A division of a cake is *super envy-free* if every player feels all other players received strictly less than $1/n$ of the total value of the cake. Clearly super envy-free implies envy-free. Barbanel [1] proved that super envy-free divisions exist if and only if the player measures are linearly independent!

This result shows only existence. Robertson and Webb show how to *construct* a super envy-free division by using a set of pieces that "witness" the linear independence of the measures.

Graph Theory Hall's famous marriage theorem can be used to develop a discrete proportional cake-cutting algorithm, as Robertson and Webb show.

Given pieces of cake $X_1, X_2, \ldots, X_n$ to be assigned to players $P_1, P_2, \ldots, P_n$, we may construct a bipartite graph G with the players and objects as vertices; there is an edge between P_i and X_j if P_i thinks X_j is at least $1/n$ of the cake. A perfect matching in G would yield an assignment of pieces to players that is proportionally fair. In this context, Hall's theorem says that there exists a perfect matching if and only if every subset S of pieces satisfies $N(S) \geq |S|$, where $N(S)$ denotes the number of players who find at least one piece in S acceptable and $|S|$ denotes the size of S.

We can ensure that such subsets of size 1 exist by letting P_1 cut the cake into n pieces equally desirable to her. Using those pieces, Robertson and Webb show that there exists a smallest subset of pieces T such that $N(T) = |T| \geq 1$ but all non-empty proper subsets $S \subset T$ satisfy the strict inequality $N(S) > |S|$. So apply Hall's theorem to assign pieces in T to the corresponding $N(T)$ players. The remaining players (if there are any) do not desire pieces in T and so are happy to give them away; they can now lump the unassigned pieces together and divide the lump fairly among themselves by appealing to induction. This process can be converted to a procedure using constructive matching algorithms.

Complexity Robertson and Webb survey several algorithms for proportional and envy-free division and compare the number of required cuts. Brams and Taylor [2] and later Robertson and Webb [7] gave procedures for envy-free division for n players. Though finite, both these procedures are unbounded, meaning they could take arbitrarily long to resolve, depending on player preferences. Whether there exists a finite, *bounded* procedure for envy-free cake division is a major open question.

For proportional division, there do exist bounded procedures for n players; the "divide-and-conquer" algorithm due to Even and Paz takes $O(n \log n)$ cuts and is the most efficient currently known for general n. Is this best possible? For specific values of n, other algorithms are more efficient than the Even-Paz algorithm, and limited evidence suggests that $O(n)$ cuts may be sufficient. The question is interesting because we know from existence theorems that $n-1$ cuts are necessary and sufficient; but what is the minimal number of cuts guaranteed by a *procedure*?

Robertson and Webb tabulate known values for this and the dual question of how much cake can be guaranteed each player by a limited number of cuts. Readers may enjoy the challenge of improving these bounds for specific values of n.

Combinatorial Topology Known n-person envy-free procedures involve a large number of cuts and would, practically speaking, decimate the cake even for the case of 4 players.

A new development [11], not discussed in the Robertson/Webb book, is the use of theorems from combinatorial topology to provide *approximate* solutions in which the number of cuts is minimal. For example, Sperner's lemma concerns triangulations of n-balls whose vertices are labelled by $n+1$ labels. It states that under appropriate boundary conditions, at least one n-simplex in the triangulation possesses all labels. More importantly, a constructive proof of Sperner's lemma surfaced in the 1960s and led to some startling constructive methods for finding fixed points [12], since the lemma is equivalent to the Brouwer fixed point theorem. Sperner's lemma relates to cake division via an extra ingredient not present in the fixed point connection—the existence of an "owner-labelling" of a triangulated ball—which allows for the accommodation of the preferences of n different players. Moreover, the constructive proof of Sperner's lemma can be converted into an algorithm for approximate envy-free division.

Related combinatorial theorems, such as Tucker's lemma, and their constructive proofs can be used to generate new algorithms for other fair division problems. The exciting news is that the connection also runs the other way—fair division questions now motivate the development of new combinatorial theorems; see [9].

Robertson and Webb describe many more connections between fair division questions and other areas of mathematics.

Cake-Cutting Algorithms is accessible to a wide mathematical audience. The language is relaxed and informal, sometimes to the point of being chatty. However, the friendly style serves to engage the uninitiated reader, and students will find it a welcome difference from stiff textbook prose. The book is effectively divided into two parts: chapters 1-6, which assume a minimal mathematical background, and chapters 8-11, which are quite technical and require much more mathematical maturity. These are separated by chapter 7, which summarizes the known cake-cutting algorithms.

By careful selection of topics, one could use material in the first seven chapters to supplement a math for liberal arts course or a topical freshman seminar, since the subject of fair division lends itself well to both mathematical excursions and philosophical discussions about "fairness". At the other end of the spectrum, advanced readers will find some important theoretical results in the final chapters that are primarily the work of the authors: impossibility theorems showing that certain kinds of divisions cannot be achieved by finite procedures, some very delicate arguments about what can be accomplished with a limited number of cuts, and the use of Ramsey partitions to address the problem of division into unequal shares.

One of the best features of the Robertson-Webb book is the variety of questions that it poses at all levels. There are exercises and projects at the end of each chapter as well as questions scattered throughout the text. Some are suitable for undergraduate projects, while others are tantalizing open problems that may take years to resolve.

For the past two years, I have had my undergraduate research students focus on fair division questions. I am happy to report that the Robertson-Webb text provided mathematical background as well as many launching points for their research; they pursued problems in envy-free *chore division* (in which players desire the smallest, rather than largest, piece) [5], examined cake division into unequal ratios, and implemented fair division algorithms on the Internet [6].

In summary, *Cake-Cutting Algorithms* will engage and challenge both veteran and novice mathematicians. Those who wish to develop their own approaches to the subject will find the book an indispensable guide to past work on the problem of cake-cutting. Mathematicians in other fields may find that the book stimulates them to think about the connections between their own work and contemporary problems in the social sciences.

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