20.2: Stokes’ Theorem

Stokes’ Theorem: Suppose that $S$ is a smooth oriented surface, with oriented boundary $C$, and $\vec{F}$ is a vector field defined in $S$. Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \text{curl} \vec{F} \, dA.$$ 

That is, the circulation of $\vec{F}$ around the curve $C$ is the same as the integral over the surface $S$ of the curl of $\vec{F}$.

Problem 1. Compute the following flux integral in two ways, 1. directly from the definition and 2. using Stokes’ theorem:

$$\int_C \vec{F} \cdot d\vec{r},$$

where $\vec{F} = (x+z)\vec{i} + x\vec{j} + y\vec{k}$ and $C$ is the upper half of the circle $x^2 + z^2 = 9$ in the plane $y = 0$, together with the $x$-axis from $(3, 0, 0)$ to $(-3, 0, 0)$, traversed counterclockwise.

Problem 2. The figure below shows an open cylindrical can, $S$, standing on the $xy$-plane. ($S$ has a bottom and sides, but no top.)

(a) Give equation(s) for the rim, $C$.

(b) If $S$ is oriented outward and downward, find $\int_S \text{curl}(-y\vec{i} + x\vec{j} + z\vec{k}) \cdot d\vec{A}$. 

![Diagram of a cylindrical can with equation $x^2 + y^2 = 9$ and direction of circulation]
**Problem 3.** Evaluate the circulation of \( \vec{G} = xy\vec{i} + z\vec{j} + 3y\vec{k} \) around a square of side 6, centered at the origin, lying in the \( yz \)-plane, and oriented counterclockwise viewed from the positive \( x \)-axis.

**Problem 4.** Find curl\((x^3\vec{i} + \sin(y^3)\vec{j} + e^{z^3}\vec{k})\). What can you say about \( \int_C (x^3\vec{i} + \sin(y^3)\vec{j} + e^{z^3}\vec{k}) \cdot d\vec{r} \) for any closed curve \( C \)?

**Challenge Problem 1.** Let \( f(x, y, z) \) and \( g(x, y, z) \) be two functions and let \( C \) be any loop (closed curve) in 3d. Show that
\[
\int_C (g \text{ grad } f + f \text{ grad } g) \cdot d\vec{r} = 0
\]
in two ways: 1) using the fundamental theorem for line integrals and 2) using Stokes’ theorem.