K3 Surfaces with $S_4$ Symmetry

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K3 Surfaces with $S_4$ Symmetry

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Outline

K3 Surfaces and Picard-Fuchs Equations

Hypersurfaces in Toric Varieties

Symmetric Families

Computing Picard-Fuchs Equations

Modular Properties

References
K3 surfaces

K3 surfaces are named after Kummer, Kähler, Kodaira . . .

and the mountain K2.
Examples of K3 surfaces

All K3 surfaces are diffeomorphic.

- Smooth quartics in $\mathbb{P}^3$
- Double covers of $\mathbb{P}^2$ branched over a smooth sextic
  \[ w^2 = f_6(x, y, z) \]
- Hypersurfaces in certain 3-dimensional toric varieties
K3 surfaces from elliptic curves

Let $E_1$ and $E_2$ be elliptic curves, and let $A = E_1 \times E_2$.

- The Kummer surface $Km(A)$ is the minimal resolution of $A/\{\pm 1\}$.
- The Shioda-Inose surface $SI(A)$ is the minimal resolution of $Km(A)/\beta$, where $\beta$ is an appropriately chosen involution.
The Hodge diamond of a K3 surface

Any K3 surface $X$ admits a nowhere-vanishing **holomorphic two-form** $\omega$ which is unique up to scalar multiples.
The Picard group

$$\text{Pic}(X) = H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z})$$

$$0 \leq \text{rank Pic}(X) \leq 20$$
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- We may identify \( \text{Pic}(X) \) with the Néron-Severi group of algebraic curves using Poincaré duality.
- \( \text{Pic}(X) \subset \omega^\perp \)
- \( \text{rank Pic}(X) \) can jump within a family of K3 surfaces
Varying complex structure for K3 surfaces

Let $X_\alpha$ be a family of K3 surfaces, and let $M$ be a free abelian group. Suppose

$$M \hookrightarrow \text{Pic}(X_\alpha).$$

Then:

- $\omega \perp M$ for each $X_\alpha$
- If $M$ has rank 19, then the variation of complex structure has 1 degree of freedom.
Picard-Fuchs equations

- A **period** is the integral of a differential form with respect to a specified homology class.
- Periods of holomorphic forms encode the complex structure of varieties.
- The **Picard-Fuchs differential equation** of a family of varieties is a differential equation that describes the way the value of a period changes as we move through the family.
- Solutions to Picard-Fuchs equations for holomorphic forms on Calabi-Yau varieties define the **mirror map**.
Picard-Fuchs equations for rank 19 families

Let $M$ be a free abelian group of rank 19, and suppose $M \hookrightarrow \text{Pic}(X_t)$.

- The Picard-Fuchs equation is Fuchsian.
- The Picard-Fuchs equation is a rank 3 ordinary differential equation.
Symmetric Squares

The symmetric square of the differential equation

\[ a_2 \frac{\partial^2 A}{\partial t^2} + a_1 \frac{\partial A}{\partial t} + a_0 A = 0 \]

is

\[ a_2 \frac{\partial^3 A}{\partial t^3} + 3a_1 a_2 \frac{\partial^2 A}{\partial t^2} + (4a_0 a_2 + 2a_1^2 + a_2 a'_1 - a_1 a'_2) \frac{\partial A}{\partial t} + (4a_0 a_1 + 2a'_0 a_2 - 2a_0 a'_2) A = 0 \]

where primes denote derivatives with respect to \( t \).
Picard-Fuchs equations and symmetric squares

Theorem

[D00, Theorem 5] The Picard-Fuchs equation of a family of rank-19 lattice-polarized K3 surfaces can be written as the symmetric square of a second-order homogeneous linear Fuchsian differential equation.
Some Picard rank 19 families

- Hosono, Lian, Oguiso, Yau:
  \[ x + 1/x + y + 1/y + z + 1/z - \Psi = 0 \]

- Verrill:
  \[ (1 + x + xy + xyz)(1 + z + zy + zyx) = (\lambda + 4)(xyz) \]

- Narumiya-Shiga:
  \[ Y_0 + Y_1 + Y_2 + Y_3 - 4tY_4 \]
  \[ Y_0 Y_1 Y_2 Y_3 - Y_4^4 \]
Lattices

Let $N$ be a lattice isomorphic to $\mathbb{Z}^n$. The dual lattice $M$ of $N$ is given by $\text{Hom}(N, \mathbb{Z})$; it is also isomorphic to $\mathbb{Z}^n$. We write the pairing of $v \in N$ and $w \in M$ as $\langle v, w \rangle$. 
Cones

A cone in $N$ is a subset of the real vector space $N_R = N \otimes \mathbb{R}$ generated by nonnegative $\mathbb{R}$-linear combinations of a set of vectors $\{v_1, \ldots, v_m\} \subset N$. We assume that cones are strongly convex, that is, they contain no line through the origin.

Figure: Cox, Little, and Schenk
Fans

A fan $\Sigma$ consists of a finite collection of cones such that:

- Each face of a cone in the fan is also in the fan
- Any pair of cones in the fan intersects in a common face.

Figure: Cox, Little, and Schenk
Simplicial fans

We say a fan $\Sigma$ is simplicial if the generators of each cone in $\Sigma$ are linearly independent over $\mathbb{R}$. 
A lattice polytope $\Diamond$ is the convex hull of a finite set of points in a lattice. We assume that our lattice polytopes contain the origin.

**Definition**

Let $\Delta$ be a lattice polytope in $\mathbb{N}$ which contains $(0,0)$. The polar polytope $\Delta^\circ$ is the polytope in $\mathbb{M}$ given by:

\[
\{(m_1, \ldots, m_k) : (n_1, \ldots, n_k) \cdot (m_1, \ldots, m_k) \geq -1 \text{ for all } (n_1, n_2) \in \Delta\}
\]
Reflexive polytopes

Definition
A lattice polytope $\Delta$ is reflexive if $\Delta^\circ$ is also a lattice polytope.

If $\Delta$ is reflexive, $(\Delta^\circ)^\circ = \Delta$. 
Fans from polytopes

We may define a fan using a polytope in several ways:

1. Take the fan $R$ over the faces of $\diamond \subset N$.
2. Refine $R$ by using other lattice points in $\diamond$ as generators of one-dimensional cones.
3. Take the normal fan $S$ to $\diamond^\circ \subset M$. 
Toric varieties as quotients

- Let $\Sigma$ be a fan in $\mathbb{R}^n$.
- Let $\{v_1, \ldots, v_q\}$ be generators for the one-dimensional cones of $\Sigma$.
- $\Sigma$ defines an $n$-dimensional toric variety $V_\Sigma$.
- $V_\Sigma$ is the quotient of a subset $\mathbb{C}^q - Z(\Sigma)$ of $\mathbb{C}^q$ by a subgroup of $(\mathbb{C}^*)^q$.
- Each one-dimensional cone corresponds to a coordinate $z_i$ on $V_\Sigma$. 
Example

Let $R$ be the fan obtained by taking cones over the faces of $\diamond$. $Z(\Sigma)$ consists of points of the form $(0, 0, z_3, z_4)$ or $(z_1, z_2, 0, 0)$.

Figure: Polygon $\diamond$

$$V_R = (\mathbb{C}^4 - Z(\Sigma))/\sim$$

$$(z_1, z_2, z_3, z_4) \sim (\lambda_1 z_1, \lambda_1 z_2, z_3, z_4)$$

$$(z_1, z_2, z_3, z_4) \sim (z_1, z_2, \lambda_2 z_3, \lambda_2 z_4)$$

where $\lambda_1, \lambda_2 \in \mathbb{C}^*$. Thus, $V_R = \mathbb{P}^1 \times \mathbb{P}^1$. 
K3 hypersurfaces

- Let $\diamond$ be a 3-dimensional reflexive polytope, with polar polytope $\diamond^\circ$.
- Let $R$ be the fan over the faces of $\diamond$.
- Let $\Sigma$ be a simplicial refinement of $R$.
- Let $\{v_k\} \subset \diamond \cap N$ generate the one-dimensional cones of $\Sigma$.
- Let $c_x$ be complex numbers.
K3 hypersurfaces

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- Let $c_x$ be complex numbers.

The following polynomial defines a K3 surface $X$ in $V_\Sigma$:

$$f = \sum_{x \in \diamond^\circ \cap M} c_x \prod_{k=1}^{q} z_k^{\langle v_k, x \rangle + 1}$$
Quasismooth and regular hypersurfaces

Let $\Sigma$ be a simplicial fan, and let $X$ be a hypersurface in $V_\Sigma$. Suppose that $X$ is described by a polynomial $f$ in homogeneous coordinates.

**Definition**
If the derivatives $\partial f / \partial z_i$, $i = 1 \ldots q$ do not vanish simultaneously on $X$, we say $X$ is quasismooth.
Quasismooth and regular hypersurfaces

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**Definition**
If the products $z_i \partial f / \partial z_i$, $i = 1 \ldots q$ do not vanish simultaneously on $X$, we say $X$ is regular and $f$ is nondegenerate.
Semiample hypersurfaces

- Let $R$ be a fan over the faces of a reflexive polytope
- Let $\Sigma$ be a refinement of $R$
- We have a proper birational morphism $\pi : V_\Sigma \to V_R$
- Let $Y$ be an ample divisor in $V_R$, and suppose $X = \pi^*(Y)$

Then $X$ is \textit{semiample}:

\textbf{Definition}

We say that a Cartier divisor $D$ is \textit{semiample} if $D$ is generated by global sections and the intersection number $D^n > 0$. 
Toric realizations of the rank 19 families

The polar polytopes $\diamondsuit^\circ$ for [HLOY04], [V96], and [NS01].

\[
f(t) = \left( \sum_{x \in \text{vertices}(\diamondsuit^\circ)} \prod_{k=1}^{q} z_k^{\langle v_k, x \rangle + 1} \right) + t \prod_{k=1}^{q} z_k.
\]
What do these polytopes have in common?

- The only lattice points of these polytopes are the vertices and the origin.
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Another symmetric polytope

Figure: The skew cube

\[ f(t) = \left( \sum_{x \in \text{vertices}(\diamondsuit)} z_{k(x)}^{\langle v_k, x \rangle + 1} \prod_{k=1}^{q} \right) + t \prod_{k=1}^{q} z_k. \]
We may view a rotation as acting either on $\diamond$ (inducing automorphisms on $X_t$) or on $\diamond^\circ$ (permuting the monomials of $f(t)$).
Symplectic Group Actions

Let $G$ be a finite group of automorphisms of a K3 surface. For $g \in G$,

$$g^*(\omega) = \rho \omega$$

where $\rho$ is a root of unity.

Definition
We say $G$ acts *symplectically* if

$$g^*(\omega) = \omega$$

for all $g \in G$. 
A subgroup of the Picard group

Definition

\[ S_G = ((H^2(X, \mathbb{Z})^G)^\perp \]

Theorem ([N80a])

\( S_G \) is a primitive, negative definite sublattice of \( \text{Pic}(X) \).
The rank of $S_G$

Lemma

- If $X$ admits a symplectic action by the permutation group $G = S_4$, then $\text{Pic}(X)$ admits a primitive sublattice $S_G$ which has rank 17.

- If $X$ admits a symplectic action by the alternating group $G = A_4$, then $\text{Pic}(X)$ admits a primitive sublattice $S_G$ which has rank 16.
Why is the Picard rank 19?

We can use the orbits of $G$ on $\diamondsuit$ to identify divisors in $(H^2(X_t, \mathbb{Z}))^G$. 
Why is the Picard rank 19?

We can use the orbits of $G$ on $\diamond$ to identify divisors in $(H^2(X_t, \mathbb{Z}))^G$.

- For the families of [HLOY04] and [V96], and the family defined by the skew cube, we conclude that $17 + 2 = 19$.
- For the family of [NS01], we conclude that $16 + 3 = 19$. 
The residue map

We will use a residue map to describe the cohomology of a K3 hypersurface $X$:

$$\text{Res} : H^3(V_\Sigma - X) \to H^2(X).$$

Anvar Mavlyutov showed that $\text{Res}$ is well-defined for quasismooth, semiample hypersurfaces in simplicial toric varieties.
Two ideals

Definition
The Jacobian ideal $J(f)$ is the ideal of $\mathbb{C}[z_1, \ldots, z_q]$ generated by the partial derivatives $\partial f / \partial z_i$, $i = 1 \ldots q$.

Definition
[BC94] The ideal $J_1(f)$ is the ideal quotient

$$\langle z_1 \partial f / \partial z_1, \ldots, z_q \partial f / \partial z_q \rangle : z_1 \cdots z_q.$$
The induced residue map

Let $\Omega_0$ be a holomorphic 3-form on $V_\Sigma$. We may represent elements of $H^3(V_\Sigma - X)$ by forms $\frac{P \Omega_0}{f^k}$, where $P$ is a polynomial in $\mathbb{C}[z_1, \ldots, z_q]$.

Mavlyutov described two induced residue maps on semiample hypersurfaces:

- $\text{Res}_J : \mathbb{C}[z_1, \ldots, z_q]/J \to H^2(X)$ is well-defined for quasismooth hypersurfaces
- $\text{Res}_{J_1} : \mathbb{C}[z_1, \ldots, z_q]/J_1 \to H^2(X)$ is well-defined for regular hypersurfaces.
Whither injectivity?

\( \text{Res}_J \) is injective for smooth hypersurfaces in \( \mathbb{P}^3 \), but this does not hold in general.

**Theorem**

[M00] If \( X \) is a regular, semiample hypersurface, then the residue map \( \text{Res}_{J_1} \) is injective.
The Griffiths-Dwork technique

Plan

We want to compute the Picard-Fuchs equation for a one-parameter family of K3 hypersurfaces $X_t$.

- Look for $\mathbb{C}(t)$-linear relationships between derivatives of periods of the holomorphic form
- Use $\text{Res}_J$ to convert to a polynomial algebra problem in $\mathbb{C}(t)[z_1, \ldots, z_q]/J(f)$
The Griffiths-Dwork technique

Procedure

1. 

\[
\frac{d}{dt} \int \text{Res} \left( \frac{P\Omega}{f^k(t)} \right) = \int \text{Res} \left( \frac{d}{dt} \left( \frac{P\Omega}{f^k(t)} \right) \right) \\
= -k \int \text{Res} \left( \frac{f'(t)P\Omega}{f^{k+1}(t)} \right)
\]
The Griffiths-Dwork technique

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2. Since \( H^*(X_t, \mathbb{C}) \) is a finite-dimensional vector space, only finitely many of the classes \( \text{Res} \left( \frac{d^j}{dt^j} \left( \frac{\Omega}{f^k(t)} \right) \right) \) can be linearly independent
The Griffiths-Dwork technique

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3. Use the reduction of pole order formula to compare classes of the form \( \text{Res} \left( \frac{P \Omega}{f^{k+1}(t)} \right) \) to classes of the form \( \text{Res} \left( \frac{Q \Omega}{f^k(t)} \right) \)
The Griffiths-Dwork technique
Implementation

Reduction of pole order

\[ \frac{\Omega_0}{f^{k+1}} \sum_i P_i \frac{\partial f}{\partial x_i} = \frac{1}{k} \frac{\Omega_0}{f^k} \sum_i \frac{\partial P_i}{\partial x_i} + \text{exact terms} \]

We use Groebner basis techniques to rewrite polynomials in terms of \( J(f) \).
The Griffiths-Dwork technique
Advantages and disadvantages

**Advantages**
We can work with arbitrary polynomial parametrizations of hypersurfaces.

**Disadvantages**
We need powerful computer algebra systems to work with $J(f)$ and $\mathbb{C}(t)[z_1, \ldots, z_q]/J(f)$.
The Skew Octahedron

- Let \( \diamond \) be the reflexive octahedron shown above.
- \( \diamond \) contains 19 lattice points.
- Let \( R \) be the fan obtained by taking cones over the faces of \( \diamond \). Then \( R \) defines a toric variety \( V_R \cong (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)/(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \).
- Consider the family of K3 surfaces \( X_t \) defined by \( f(t) = \left( \sum_{x \in \text{vertices}(\diamond)} \prod_{k=1}^{q} z_{k}^{\langle v_{k}, x \rangle + 1} \right) + t \prod_{k=1}^{q} z_{k} \).
- \( X_t \) are generally quasismooth but not regular.
The Picard-Fuchs equation

Theorem ([KLMSW10])
Let $A = \int \text{Res} \left( \frac{\Omega_0}{f} \right)$. Then $A$ is the period of a holomorphic form on $X_t$, and $A$ satisfies the Picard-Fuchs equation

$$
\frac{\partial^3 A}{\partial t^3} + \frac{6(t^2 - 32)}{t(t^2 - 64)} \frac{\partial^2 A}{\partial t^2} + \frac{7t^2 - 64}{t^2(t^2 - 64)} \frac{\partial A}{\partial t} + \frac{1}{t(t^2 - 64)} A = 0.
$$

As expected, the differential equation is third-order and Fuchsian.
Symmetric square root

The symmetric square root of our Picard-Fuchs equation is:

\[ \frac{\partial^2 A}{\partial t^2} + \frac{(2t^2 - 64)}{t(t^2 - 64)} \frac{\partial A}{\partial t} + \frac{1}{4(t^2 - 64)} A = 0. \]
Mirror Moonshine for a one-parameter family of K3 surfaces arises when there exists a genus 0 modular group \( \Gamma \subset PSL_2(\mathbb{R}) \) such that

- The Picard-Fuchs equation gives the base of the family the structure of a (pull-back of a) modular curve \( \overline{\mathbb{H}}/\Gamma \).
- The mirror map is commensurable with a hauptmodul for \( \Gamma \).
- The holomorphic solution to the Picard-Fuchs equation is a \( \Gamma \)-modular form of weight 2.
**Mirror Moonshine from geometry**

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<tr>
<th>Example</th>
<th>[HLOY04]</th>
<th>[V96]</th>
</tr>
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<tbody>
<tr>
<td>Shioda-Inose structure</td>
<td>$\text{SI}(E_1 \times E_2)$</td>
<td>$\text{SI}(E_1 \times E_2)$</td>
</tr>
<tr>
<td></td>
<td>$E_1, E_2$ are 6-isogenous</td>
<td>$E_1, E_2$ are 3-isogenous</td>
</tr>
<tr>
<td>$\text{Pic}(X)^\perp$</td>
<td>$H \oplus \langle 12 \rangle$</td>
<td>$H \oplus \langle 6 \rangle$</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>$\Gamma_0(6) + 6$</td>
<td>$\Gamma_0(6) + 3 \subset \Gamma_0(3) + 3$</td>
</tr>
</tbody>
</table>
Geometry of the skew octahedron family

- $X_t$ is a family of Kummer surfaces
- Each surface can be realized as $Km(E_t \times E_t)$
- The generic transcendental lattice is $2H \oplus \langle 4 \rangle$
The modular group

We use our symmetric square root and the table of [LW06] to show that:

\[ \Gamma = \Gamma_0(4|2) \]

= \left\{ \begin{pmatrix} a & b/2 \\ 4c & d \end{pmatrix} \in PSL_2(\mathbb{R}) \mid a, b, c, d \in \mathbb{Z} \right\}

\Gamma_0(4|2) \text{ is conjugate in } PSL_2(\mathbb{R}) \text{ to } \Gamma_0(2) \subset PSL_2(\mathbb{Z}) = \Gamma_0(1) + 1.

Doran, C. Picard-Fuchs uniformization and modularity of the mirror map. *Communications in Mathematical Physics* 212 (2000), no. 3, 625–647.


http://people.brandeis.edu/~lian/Schiff.pdf


The next big polytope . . .