Mirror Symmetry Through Reflexive Polytopes
Physical and Mathematical Dualities

Ursula Whitcher

Harvey Mudd College

April 2010
Outline

String Theory and Mirror Symmetry

Some Complex Geometry

Reflexive Polytopes

From Polytopes to Spaces
Where’s the Theory of Everything?

▶ We understand gravity on a large spatial scale (planets, stars, galaxies).

▶ We understand quantum physics on a small spatial scale (electrons, photons, quarks).

Figure: S. Bush et al.
Are Strings the Answer?

- “Fundamental” particles are strings vibrating at different frequencies.

- Strings wrap other dimensions!
T-Duality

Pairs of Universes
An extra dimension shaped like a circle of radius $R$ and an extra dimension shaped like a circle of radius $\alpha'/R$ yield indistinguishable physics! (The slope parameter $\alpha'$ has units of length squared.)

Figure: Large radius, few windings

Figure: Small radius, many windings
Heterotic String Theory

Supersymmetries
Interchange bosons (spin $n$) with fermions (spin $n + \frac{1}{2}$)

Hybrid Vigor

- Equations of motion split into “left-moving” and “right-moving” solutions.
- Operators $Q$ and $\bar{Q}$ act on left-moving and right-moving states, respectively.
- $Q$ and $\bar{Q}$ are well-defined up to sign.
Building a Model

Locally, space-time should look like

\[ M_{3,1} \times V. \]

- \( M_{3,1} \) is four-dimensional space-time
- \( V \) is a \( d \)-dimensional complex manifold
- Physicists require \( d = 3 \) (6 real dimensions)
- \( V \) is a Calabi-Yau manifold
- For differential geometers, \( V \) is a Ricci-flat Kähler-Einstein manifold
- Our choice of geometric model fixes a choice of sign for the operators \( Q \) and \( \bar{Q} \)
Geometry Appears

Physical properties of our model for the universe should correspond to geometric properties of the space $V$.

We will be particularly interested in the complex moduli and Kähler moduli of $V$. 
The Heterotic Duality

- Changing the sign of the operator $Q$ corresponds to choosing a new Calabi-Yau manifold $V^\circ$.
- The geometrical properties of $V$ and $V^\circ$ yield equivalent physical theories.

Mirror symmetry is the study of the mathematical consequences of this duality.
Realizing Mirror Symmetry Geometrically

We need:

- Complex manifolds
- which are Calabi-Yau
- and arise in paired or “mirror” families
- with dual geometric properties.

Varying complex structure in one family should correspond to varying Kähler structure in the other family.
Complex Structure

An $n$-dimensional complex manifold is a geometric space which looks locally like $\mathbb{C}^n$.

Example: Elliptic Curves

We can think of varying the parameter $\tau$ as either changing the complex manifold, or changing the complex structure on an underlying topological 2-torus.
We can pair vectors \( v \) and \( w \) with their tails at a point \( z \) in \( \mathbb{C} \) using the product for complex numbers:

\[
\langle v, w \rangle = v \overline{w}
\]

Note that \( \langle v, v \rangle = \|v\|^2 \).

More generally, if \( \vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \) and \( \vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \) are vectors with their tails at a point \( z = (z_1, \ldots, z_n) \) in \( \mathbb{C}^n \), their standard Hermitian product is given by

\[
\langle \vec{v}, \vec{w} \rangle = \sum v_i \overline{w_i}
\]

\[
= \vec{v}^T \overline{\vec{w}}
\]
Kähler Structure

Hermitian metrics

A Hermitian metric $H$ tells us how to pair tangent vectors at any point of a complex manifold and obtain a complex number.

$$H(\vec{v}, \vec{w}) = \overline{H(\vec{w}, \vec{v})}$$
Kähler Structure

Hermitian metrics

A **Hermitian metric** $H$ tells us how to pair tangent vectors at any point of a complex manifold and obtain a complex number.

$$ H(\vec{v}, \vec{w}) = \overline{H(\vec{w}, \vec{v})} $$

Elliptic Curve Example

We can use the standard Hermitian product on $\mathbb{C}$ to describe a Hermitian metric for tangent vectors to an elliptic curve.
A Kähler metric is a special type of Hermitian metric which can be written in local coordinates as follows. If $\vec{v}$ and $\vec{w}$ are vectors with their tails at a point $z_0$ in $\mathbb{C}^n$,

$$\kappa(\vec{v}, \vec{w}) = \vec{v}^T (I + G(z)) \vec{w}.$$ 

Here $I$ is the identity matrix and

$$G(z) = \begin{pmatrix}
g_{11}(z) & \cdots & g_{1n}(z) \\
\vdots & \ddots & \vdots \\
g_{n1}(z) & \cdots & g_{nn}(z)
\end{pmatrix}$$

vanishes up to order 2 at $z_0$. 

We need:

- Complex manifolds $V$
- which are Calabi-Yau
- and arise in paired or “mirror” families
- with dual geometric properties.

Varying complex structure in one family should correspond to varying Kähler structure in the other family.
Batyrev’s Insight

We can describe mirror families of Calabi-Yau manifolds using combinatorial objects called reflexive polytopes.
Lattice Polygons

The points in the plane with integer coordinates form a lattice \( N \). A lattice polygon is a polygon in the plane which has vertices in the lattice.
Reflexive Polygons

We say a lattice polygon is reflexive if it has only one lattice point, the origin, in its interior.

Figure: A reflexive triangle
Describing a Reflexive Polygon

- List the vertices

\[(0, 1), (1, 0), (-1, -1)\]

- List the equations of the edges

\[-x - y = -1, 2x - y = -1, -x + 2y = -1\]
Describing a Reflexive Polygon

- List the vertices
  
  \( \{(0, 1), (1, 0), (-1, -1)\} \)
Describing a Reflexive Polygon

- List the vertices

\[ \{(0, 1), (1, 0), (-1, -1)\} \]

- List the equations of the edges
Describing a Reflexive Polygon

- List the vertices

\{ (0, 1), (1, 0), (−1, −1) \}

- List the equations of the edges

\[ −x − y = −1 \]
\[ 2x − y = −1 \]
\[ −x + 2y = −1 \]
A Dual Lattice

Let $M$ be another copy of the points in the plane with integer coordinates. The dot product lets us pair points in $N$ with points in $M$:

$$(n_1, n_2) \cdot (m_1, m_2) = n_1 m_1 + n_2 m_2$$
Let $\Delta$ be a lattice polygon in $\mathbb{N}$ which contains $(0,0)$. The polar polygon $\Delta^\circ$ is the polygon in $\mathbb{M}$ given by:

$$\{(m_1, m_2) : (n_1, n_2) \cdot (m_1, m_2) \geq -1 \text{ for all } (n_1, n_2) \in \Delta\}$$
Polar Polygons

Edge equations define new polygons

Let \( \Delta \) be a lattice polygon in \( \mathbb{N} \) which contains \((0,0)\). The polar polygon \( \Delta^\circ \) is the polygon in \( \mathbb{M} \) given by:

\[
\{(m_1, m_2) : (n_1, n_2) \cdot (m_1, m_2) \geq -1 \text{ for all } (n_1, n_2) \in \Delta\}
\]

\[
(x, y) \cdot (-1, -1) = -1
\]
\[
(x, y) \cdot (2, -1) = -1
\]
\[
(x, y) \cdot (-1, 2) = -1
\]
Polar Polygons

Edge equations define new polygons

Let $\Delta$ be a lattice polygon in $\mathbb{N}$ which contains $(0, 0)$. The polar polygon $\Delta^\circ$ is the polygon in $\mathbb{M}$ given by:

$$\{(m_1, m_2) : (n_1, n_2) \cdot (m_1, m_2) \geq -1 \text{ for all } (n_1, n_2) \in \Delta\}$$

$$(x, y) \cdot (-1, -1) = -1$$
$$(x, y) \cdot (2, -1) = -1$$
$$(x, y) \cdot (-1, 2) = -1$$

Figure: Our triangle’s polar polygon
Mirror Pairs

If $\Delta$ is a reflexive polygon, then:

- $\Delta^\circ$ is also a reflexive polygon
- $(\Delta^\circ)^\circ = \Delta$

$\Delta$ and $\Delta^\circ$ are a mirror pair.
A Polygon Duality

Mirror pair of triangles

Figure: 3 boundary lattice points

Figure: 9 boundary lattice points

\[3 + 9 = 12\]
Mirror Pairs of Polygons

Figure: F. Rohsiepe, “Elliptic Toric K3 Surfaces and Gauge Algebras”
Other Dimensions

Definition
Let \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_q \} \) be a set of points in \( \mathbb{R}^k \). The polytope with vertices \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_q \} \) is the convex hull of these points.
Polar Polytopes

Let \( \mathbb{N} \) be the lattice of points with integer coordinates in \( \mathbb{R}^k \). A lattice polytope has vertices in \( \mathbb{N} \).

As before, we have a dual lattice \( \mathbb{M} \) and a dot product

\[
(n_1, \ldots, n_k) \cdot (m_1, \ldots, m_k) = n_1 m_1 + \cdots + n_k m_k
\]

Definition

Let \( \Delta \) be a lattice polygon in \( \mathbb{N} \) which contains \( (0, 0) \). The polar polytope \( \Delta^\circ \) is the polytope in \( \mathbb{M} \) given by:

\[
\{(m_1, \ldots, m_k) : (n_1, \ldots, n_k) \cdot (m_1, \ldots, m_k) \geq -1 \text{ for all } (n_1, n_2) \in \Delta\}
\]
Reflexive Polytopes

Definition
A lattice polytope $\Delta$ is reflexive if $\Delta^\circ$ is also a lattice polytope.

- If $\Delta$ is reflexive, $(\Delta^\circ)^\circ = \Delta$.
- $\Delta$ and $\Delta^\circ$ are a mirror pair.
Mirror Polytopes Yield Mirror Spaces

polytope $\leftrightarrow$ polar polytope

Laurent polynomial $\leftrightarrow$ mirror Laurent polynomial

space $\leftrightarrow$ mirror space
From Polytopes to Polynomials

- Standard basis vectors in $N$ ↔ variables $z_i$

  \[
  (1, 0, \ldots, 0) \leftrightarrow z_1 \\
  (0, 1, \ldots, 0) \leftrightarrow z_2 \\
  \ldots \\
  (0, 0, \ldots, 1) \leftrightarrow z_n
  \]

- Lattice points in $\Delta^\circ$ ↔ monomials defined on $(\mathbb{C}^*)^n$

  \[
  (m_1, \ldots, m_k) \leftrightarrow \\
  z_1^{(1, 0, \ldots, 0) \cdot (m_1, \ldots, m_k)} z_2^{(0, 1, \ldots, 0) \cdot (m_1, \ldots, m_k)} \ldots z_k^{(0, 0, \ldots, 1) \cdot (m_1, \ldots, m_k)}
  \]

- $\Delta^\circ$ ↔ Laurent polynomials $p_\alpha$ defined on $(\mathbb{C}^*)^n$
From Polynomials to Spaces

The solutions to the Laurent polynomials $p_\alpha$ describe geometric spaces.

- $-z_1^{-1} + z_1 = 0$
- $z_1^{-1} + z_1 = 0$
- $z_1^5 + z_2^5 + z_3^5 + z_4^5 + 1 = 0$

*Figure:* Slice of a Calabi-Yau threefold
Our Laurent polynomials \( p_\alpha \) define spaces which are not compact: \( |z_i| \) can be infinitely large. We can solve this problem by adding in some “points at infinity” using a standard procedure from algebraic geometry.

The resulting compact spaces \( V_\alpha \) are Calabi-Yau varieties of dimension \( d = k - 1 \).

- When \( k = 2 \), for generic choice of \( \alpha \), the \( V_\alpha \) are elliptic curves.
- When \( k = 4 \), for generic choice of \( \alpha \), the \( V_\alpha \) are smooth 3-dimensional Calabi-Yau manifolds.
Mirror Symmetry

polytope $\leftrightarrow$ polar polytope

$\downarrow$

Laurent polynomials $p_\alpha$ $\leftrightarrow$ mirror Laurent polynomials $p^{\circ}_\alpha$

$\downarrow$

spaces $V_\alpha$ $\leftrightarrow$ mirror spaces $V^{\circ}_\alpha$
Counting Complex Moduli

The possible deformations of complex structure of $V_\alpha$ form a complex vector space of dimension $h^{d-1,1}(V_\alpha)$.
For $k \geq 4$,

$$h^{d-1,1}(V_\alpha) = \ell(\Delta^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ)$$

- $\ell() = \text{number of lattice points}$
- $\ell^*(()) = \text{number of lattice points in the relative interior of a polytope or face}$
- The $\Gamma^\circ$ are codimension 1 faces of $\Delta^\circ$
- The $\Theta^\circ$ are codimension 2 faces of $\Theta^\circ$
- $\hat{\Theta}^\circ$ is the face of $\Delta$ dual to $\Theta^\circ$
For $k \geq 4$, 

$$h^{1,1}(V_\alpha) = \ell(\Delta) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta})$$

- $\ell()$ = number of lattice points
- $\ell^*(())$ = number of lattice points in the relative interior of a polytope or face
- The $\Gamma$ are codimension 1 faces of $\Delta$
- The $\Theta$ are codimension 2 faces of $\Theta$
- $\hat{\Theta}$ is the face of $\Delta$ dual to $\Theta$
Comparing $V$ and $V^\circ$

For $k \geq 4$,

$$h^{1,1}(V_\alpha) = \ell(\Delta) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta})$$

$$h^{d-1,1}(V_\alpha) = \ell(\Delta^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ)$$
Comparing $V$ and $V^\circ$

For $k \geq 4$,

$$h^{1,1}(V_\alpha) = \ell(\Delta) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta})$$

$$h^{d-1,1}(V_\alpha) = \ell(\Delta^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ)$$

$$h^{1,1}(V_\alpha^\circ) = \ell(\Delta^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ)$$

$$h^{d-1,1}(V_\alpha^\circ) = \ell(\Delta) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta})$$
We have mirror families of Calabi-Yau varieties $V_\alpha$ and $V^\circ_\alpha$ of dimension $d = k - 1$.

\[
h^{1,1}(V_\alpha) = h^{d-1,1}(V^\circ_\alpha)
\]
\[
h^{d-1,1}(V_\alpha) = h^{1,1}(V^\circ_\alpha)
\]
An Example

Four-dimensional analogue:

- $\Delta$ has vertices $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, and $(-1, -1, -1, -1)$.
- $\Delta^\circ$ has vertices $(-1, -1, -1, -1)$, $(4, -1, -1, -1)$, $(-1, 4, -1, -1)$, $(-1, -1, 4, -1)$, and $(-1, -1, -1, 4)$.
An Example

Four-dimensional analogue:

- $\Delta$ has vertices $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, and $(-1, -1, -1, -1)$.
- $\Delta^\circ$ has vertices $(-1, -1, -1, -1)$, $(4, -1, -1, -1)$, $(-1, 4, -1, -1)$, $(-1, -1, 4, -1)$, and $(-1, -1, -1, 4)$.

\[
h^{1,1}(V_\alpha) = \ell(\Delta) - n - 1 - \sum_\Gamma \ell^*(\Gamma) + \sum_\Theta \ell^*(\Theta)\ell^*(\hat{\Theta})
\]

\[= 6 - 4 - 1 - 0 - 0 = 1.\]
Example (Continued)

- $\Delta$ has vertices $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$, and $(-1, -1, -1, -1)$.
- $\Delta^\circ$ has vertices $(-1, -1, -1, -1), (4, -1, -1, -1), (-1, 4, -1, -1), (-1, -1, 4, -1)$, and $(-1, -1, -1, 4)$.

\[
h^{1,1}(V_\alpha) = 1
\]

\[
h^{3-1,1}(V_\alpha) = \ell(\Delta^\circ) - n - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ) = 126 - 4 - 1 - 20 - 0 = 101.
\]
The Hodge Diamond
Calabi-Yau Threefolds

\[
\begin{array}{cccccc}
0 & 0 & 1 & 0 & 0 & 0 \\
1 & h^{2,1}(V) & h^{1,1}(V) & h^{2,1}(V) & 1 \\
0 & h^{1,1}(V) & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
\end{array}
\]
The Hodge Diamond
Calabi-Yau Threefolds

\[
\begin{array}{cccc}
& & 1 & \\
& 0 & & \\
0 & & h^{1,1}(V) & 0 \\
& 1 & h^{2,1}(V) & \\
& & 0 & \\
1 & & 0 & 1
\end{array}
\]

\[
\begin{align*}
V_\alpha & =
\begin{array}{cccc}
1 & & & \\
0 & & & \\
0 & & & \\
1 & & & 1
\end{array} \\
V^{\circ}_\alpha & =
\begin{array}{cccc}
1 & & & \\
0 & & & \\
0 & & & \\
1 & & & 1
\end{array}
\end{align*}
\]