Informally, a vector bundle associates a vector space with each point of another space. Vector bundles may be constructed over general topological spaces; we will only examine those that have the additional structure of an algebraic variety.

1 Construction of a vector bundle

Fixing a field $k$, the affine variety $\mathbb{A}^n_k$ together with a point designated as the origin has a structure of the vector space $k^n$. Because we want to treat our vector spaces as algebraic varieties, each vector space is embedded as an affine variety inside another variety referred to as the total space, and is attached to the base space by means of a morphism of varieties.

Definition. A family of vector spaces over an algebraic variety $X$ is a surjective morphism of varieties $\pi : E \to X$, together with a fixed vector space structure over every fiber $E_x = \pi^{-1}(x)$:

$$E_x \cong \mathbb{A}^n_k \hookrightarrow k^n.$$  

It is usually required that the dimension $n$ of the vector space be constant over all fibers and finite.

Definition. A morphism between the two families of vector spaces $\pi : E \to X$ and $\rho : F \to X$ is given by a commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\varphi} & F \\
\downarrow{\pi} & & \downarrow{\rho} \\
X & & \\
\end{array}
$$
where the morphism of varieties \( \varphi : E \to F \), when restricted to any fiber \( E_x (x \in X) \), is required to be a linear map of \( E_x \) into \( F_x \) with respect to their vector space structures. Furthermore, if \( \varphi \) gives an isomorphism of vector spaces between \( E_x \) and \( F_x \) for all \( x \in X \), then the diagram is an isomorphism.

Any family of vector spaces \( \pi : E \to X \) may be restricted to an open subset \( U \subset X \). Then the restriction
\[
\pi|_U : \pi^{-1}(U) \to U
\]
is also a family of vector spaces.

**Definition.** The trivial bundle, a particularly nice family of vector spaces, is simply the direct product of a vector space and the variety together with the projection onto the variety:
\[
E = X \times \mathbb{A}^n_k, \quad \pi : E \to X, \quad \pi : (x, v) \mapsto x.
\]

In taking the direct product of two varieties, we should discuss what sort of varieties we are working with. We want \( X \) to possibly be projective, and \( \mathbb{A}^n_k \) is affine. In the absence of the full generality of schemes, we can take our varieties to be quasiprojective, that is a closed subset of an open subset of a projective space \( \mathbb{P}^n_k \). Notice that the class of quasiprojective varieties includes both affine and projective varieties. The direct product of two quasiprojective varieties, defined via the Segre embedding into a projective space, can be shown to be again a quasiprojective variety.

**Definition.** A vector bundle is a locally trivial family of vector spaces. More precisely, a family \( \pi : E \to X \) is a vector bundle if there exists an open cover \( X = \bigcup U_\alpha \) such that the restriction to each \( U_\alpha \) is isomorphic to a trivial bundle via a diagram
\[
\begin{array}{ccc}
\pi^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times \mathbb{A}^n_k \\
\pi \downarrow & & \downarrow \rho \\
U_\alpha & \xrightarrow{} & \\
\end{array}
\]
where \( \rho \) is the projection to the first coordinate and \( \varphi_\alpha \) is a morphism of varieties giving vector space isomorphisms over fibers of \( \pi \) and \( \rho \).

In an abuse of notation, we sometimes conflate the vector bundle and the total space, \( E \).
The (constant) dimension $n$ of the vector spaces is the rank of the vector bundle. A rank-1 vector bundle is referred to as a line bundle, for the obvious reason.

Fixing an open cover $\{U_\alpha\}$ and set of isomorphisms $\{\varphi_\alpha\}$, we can think of each $\varphi_\alpha$ as giving coordinate systems for the fibers $E_x (x \in U_\alpha)$, mapping them into $\mathbb{A}_k^n$. For any nonempty intersection $V = U_\alpha \cap U_\beta$, the diagram

\[
\begin{array}{ccc}
V \times \mathbb{A}_k^n & \xrightarrow{\varphi_\alpha} & \pi^{-1}(V) \\
\downarrow{\pi} & & \downarrow{\varphi_\beta} \\
V & \xrightarrow{\varphi_\alpha^{-1}} & V \times \mathbb{A}_k^n
\end{array}
\]

indicates that we have the change of coordinates

\[
\{x\} \times \mathbb{A}_k^n \xrightarrow{\varphi_\alpha^{-1}} E_x \xrightarrow{\varphi_\beta} \{x\} \times \mathbb{A}_k^n
\]

between any two coordinatizations of a fiber. It can be shown that the matrix of the change of coordinates has entries given by regular functions on the intersection $V$.

2 Example: the tautological bundle over $\mathbb{P}^n_k$

Each point of a projective space is associated with a one-dimensional subspace of an affine space; this suggests a particular line bundle, known as the tautological bundle.

Given the projective space $\mathbb{P}^n_k$, our total space $E$ is a subset of $\mathbb{P}^n_k \times \mathbb{A}^{n+1}_k$, where

\[
E = \{(p, v) \mid p \in \mathbb{P}^n_k, v \in \mathbb{A}_k^n\},
\]

and $l_p$ denotes the one-dimensional subspace of $\mathbb{A}_k^{n+1}$ associated with $p$. The morphism $\pi : E \to \mathbb{P}^n_k$ is simply the projection to the first coordinate. To show that this is a vector bundle, it is sufficient to construct a local trivialization of $E$.

Fix a coordinate system of $E$, where a point $(p, v) \in E$ is

\[
(p, v) = ((p_0 : \cdots : p_n), (v_0, \cdots, v_n)) ,
\]

and the $p_i$ are homogeneous coordinates. Then $\mathbb{P}^n_k$ has the standard open cover $\{U_i\}$, where $U_i$ is the affine subset with $p_i \neq 0$. For each $U_i$, we want a morphism $\varphi_i$ such that the diagram
The construction is straightforward. Define $\varphi_i$ as

$$\varphi_i : (p, v) \mapsto (p, v_i)$$

$$\varphi_i^{-1} : (p, v_i) \mapsto (p, (v_i/p_i)p).$$

Over a fiber of a given point $p \in \mathbb{P}_k^n$, it is clear that $\varphi_i$ is a linear map. Also, because the affine point $v$ is a scalar multiple of $p$ by construction and $p_i \neq 0$, the inverse $\varphi_i^{-1}$ recovers $v$. Finally, the projective component $p$ is preserved by all morphisms, so the diagram commutes.

### 3 Sections of a vector bundle

A vector bundle is often viewed as a sheaf of sections.

**Definition.** A *section* of a vector bundle $\pi : E \to X$ is a morphism $s : X \to E$ such that $\pi \circ s = \text{id}$.

One can visualize a section as a slice of the total space for which the projection $\pi$ is bijective.

The set of all sections over an open subset $U \subset X$ is denoted $\mathcal{L}_E(U)$. Any two sections $s_1, s_2 \in \mathcal{L}_E(U)$ admit a sum

$$(s_1 + s_2)(x) = s_1(x) + s_2(x),$$

and a section $s \in \mathcal{L}_E(U)$ admits scaling by a regular function $f$ on $U$

$$(fs)(x) = f(x)s(x)$$

where $f(x) \in k$ acts as on the vector space $E_x$. Then $\mathcal{L}_E(U)$ is a module over the regular functions on $U$, $\mathcal{O}_X(U)$.

In fact, $\mathcal{L}_E$ is a sheaf of modules over the structure sheaf $\mathcal{O}_X$. Furthermore, it can be shown that there is a one-to-one correspondence of vector bundles over $X$ and locally free sheaves of $\mathcal{O}_X$-modules with finite rank, i.e. sheaves $\mathcal{L}$ for which $\mathcal{L}(U_\alpha) \cong \bigoplus_{i=1}^n \mathcal{O}_X$ on some open cover $U_\alpha$; we will not prove this here. This formulation of vector bundles makes a number of constructions easier, such as direct sums, tensor products, and duals of bundles.
4 The tangent bundle and differential forms

Recall that given an affine smooth variety $X \subset \mathbb{A}^n_k$, each point $p \in X$ has a tangent space $T_pX$ defined as

$$T_pX = \{ v \in \mathbb{A}^n_k \mid (d_p f)(v) = 0 \text{ for all } f \in I(X) \},$$

where the differential of a polynomial $f$ at $p \in X$ is the function on $\mathbb{A}^n_k$

$$(d_p f)(x) = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \right)(p) (x_i - p_i).$$

The value of a differential $(d_p f)(x)$ is a polynomial in the coordinates of both $p$ and $x$, so the space $E = \{(p, x) \mid x \in T_pX \} \subset X \times \mathbb{A}^n_k$

is an affine variety defined by the polynomials $\bigcup \{ f, df \}$ for $f \in I(X)$, and is a vector bundle known as the tangent bundle. Verifying that $E$ is locally trivial is a little tricky (and is easier with the ‘sheaf of modules’ formulation of a vector bundle), so we do not do it here. The idea is that the tangent space at a point is a local property.

The dual of the tangent bundle is the cotangent bundle $\Omega$, the $m^{th}$ exterior power $\Omega^m$ is the bundle of $m$-dimensional differential forms, and the highest exterior power $\Omega^n$ is a line bundle known as the canonical bundle. These vector bundles are of particular importance as they turn out to be intrinsically defined, i.e. they do not depend on any particular embedding.

References
